Lecture Five: The Cacciopoli Inequality

1 The Cacciopoli Inequality

The Cacciopoli (or Reverse Poincare) Inequality bounds similar terms to the Poincare inequalities studied last time, but the other way around. The statement is this.

**Theorem 1.1** Let \( u : B_{2r} \to \mathbb{R} \) satisfy \( u \triangle u \geq 0 \). Then

\[
\int_{B_r} |\nabla u|^2 \leq \frac{4}{r^2} \int_{B_{2r}\setminus B_r} u^2. 
\]  

(1)

First prove a Lemma.

**Lemma 1.2** If \( u : B_{2r} \to \mathbb{R} \) satisfies \( u \triangle u \geq 0 \), and \( \phi : B_{2r} \to \mathbb{R} \) is non-negative with \( \phi = 0 \) on \( \partial B_{2r} \), then

\[
\int_{B_{2r}} \phi^2 |\nabla u|^2 \leq 4 \int_{B_{2r}} |u|^2 |\nabla \phi|^2.
\]

(2)

**Proof** Consider

\[
0 \leq \int_{B_{2r}} \phi^2 u \triangle u.
\]

(3)

Clearly \( \int_{\partial B_{2r}} \phi^2 u \nabla u \cdot dS = 0 \), so apply Stokes’ theorem to get \( \int_{B_{2r}} \phi^2 u \triangle u + \int_{B_{2r}} \nabla (\phi^2 u) \cdot \nabla u = 0 \). From this

\[
0 \leq -\int_{B_{2r}} \nabla (\phi^2 u) \nabla u = -2 \int_{B_{2r}} \phi u \nabla \phi \cdot \nabla u - \int_{B_{2r}} \phi^2 |\nabla u|^2,
\]

(4)

and so

\[
\int_{B_{2r}} \phi^2 |\nabla u|^2 \leq -2 \int_{B_{2r}} \phi u \nabla \phi \nabla u \leq 2 \int_{B_{2r}} \phi |u| |\nabla \phi| |\nabla u|.
\]

(5)

(6)
Recall the inequality $\int fg \leq \left( \int f^2 \right)^{1/2} \left( \int g^2 \right)^{1/2}$ for any functions $f$ and $g$ (this is one form of the Cauchy-Schwarz inequality), and apply it above to get

$$\int_{B_{2r}} \phi^2 |\nabla u|^2 \leq 2 \left( \int_{B_{2r}} \phi^2 |\nabla u|^2 \right)^{1/2} \left( \int_{B_{2r}} |u|^2 |\nabla \phi|^2 \right)^{1/2}.$$  (7)

Dividing and squaring then gives

$$\int_{B_{2r}} \phi^2 |\nabla u|^2 \leq 4 \int_{B_{2r}} |u|^2 |\nabla \phi|^2.$$  (8)

To complete the proof of theorem 1.1 pick

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq r; \\ \frac{2r-|x|}{r} & \text{if } r < x \leq 2r, \end{cases}$$

so $|\nabla \phi| = 0$ on $B_r$ and $|\nabla \phi| = 1/r$ on $B_{2r} \setminus B_r$. Substitute this into the lemma to obtain the result, namely

$$\int_{B_r} |\nabla u|^2 \leq \frac{4}{r^2} \int_{B_{2r} \setminus B_r} u^2.$$  (9)

2 Applications of the Cacciopoli Inequality

2.1 Bounding the growth of a harmonic function

One nice consequence of the Cacciopoli Inequality is the following inequality bounding the rate at which a harmonic function can decay.

Proposition 2.1 There are strictly positive dimensional constants $k(n)$ such that

$$\int_{B_{2r}} u^2 \geq (1 + k(n)) \int_{B_r} u^2$$

for all harmonic functions $u : B_{2r} \to \mathbb{R}$.

Proof Let $\phi$ be a test function as before, and consider

$$\int_{B_{2r}} |\nabla (\phi u)|^2 = \int_{B_{2r}} |\phi \nabla u + u \nabla \phi|^2$$

$$= \int_{B_{2r}} \phi^2 |\nabla u|^2 + u^2 |\nabla \phi|^2 + 2u \phi \nabla \phi \cdot \nabla u.$$

Apply Cauchy-Schwarz and lemma 1.2 to get
\[
\int_{B_{2r}} |\nabla (\phi u)|^2 \leq \int_{B_{2r}} \phi^2 |\nabla u|^2 + \int_{B_{2r}} u^2 |\nabla \phi|^2 + 2 \left( \int_{B_{2r}} \phi^2 |\nabla u|^2 \right)^{1/2} \left( \int_{B_{2r}} u^2 |\nabla \phi|^2 \right)^{1/2}
\leq 2 \int_{B_{2r}} \phi^2 |\nabla u|^2 + 2 \int_{B_{2r}} u^2 |\nabla \phi|^2.
\leq 10 \int_{B_{2r}} u^2 |\nabla \phi|^2.
\]

Now make the same choice of \( \phi \) as before to give
\[
\int_{B_{2r}} |\nabla (\phi u)|^2 \leq \frac{10}{r^2} \int_{B_{2r} \setminus B_r} u^2
\] (11)

and apply Dirichlet-Poincare to the left hand side to get
\[
\frac{1}{C(n)r^2} \int_{B_{2r}} \phi^2 u^2 \leq \frac{10}{r^2} \int_{B_{2r} \setminus B_r} u^2.
\] (12)

Since \((\phi u)^2\) is a positive function we can reduce the area of the integration, therefore
\[
k(n) \int_{B_r} \phi^2 u^2 \leq \int_{B_{2r} \setminus B_r} u^2.
\] (13)

for \(k(n) = \frac{1}{10C(n)}\). Finally note that \(\phi = 1\) on \(B_r\), so
\[
k(n) \int_{B_r} u^2 \leq \int_{B_{2r} \setminus B_r} u^2,
\] (14)

and
\[
(1 + k(n)) \int_{B_r} u^2 \leq \int_{B_{2r}} u^2.
\] (15)

This completes the proof. \(\square\)

2.2 Bounding the growth of the energy of a harmonic function

We will now prove a similar inequality for the Dirichlet energy of a harmonic function.

**Proposition 2.2** There are dimensional constants \(c(n)\) such that
\[
\int_{B_{2r}} |\nabla u|^2 \geq (1 + \theta(n)) \int_{B_r} |\nabla u|^2.
\] (16)

for all harmonic functions \(u : B_{2r} \rightarrow \mathbb{R}\).
**Proof** It suffices to show that

\[ c(n) \int_{B_r} |\nabla u|^2 \leq \int_{B_{2r}\setminus B_r} |\nabla u|^2. \]  

(17)

To do this we use two inequalities. Firstly we will state and use without proof the Neumann-Poincare inequality for an annulus, namely if \( A = \frac{1}{\text{vol}_{B_{2r}\setminus B_r}} \int_{B_{2r}\setminus B_r} u \) then

\[ \int_{B_{2r}\setminus B_r} (u - A)^2 \leq d(n)r^2 \int_{B_{2r}\setminus B_r} |\nabla u|^2. \]  

(18)

Secondly we use Cacciopoli, noting that if \( \Delta u = 0 \) then \( \Delta (u + A) = 0 \), and \( \nabla (u + A) = \nabla u \), to give

\[ r^2 \int_{B_r} |\nabla u|^2 \leq 4 \int_{B_{2r}\setminus B_r} (u - A)^2. \]  

(19)

Together (15) and (16) give

\[ \frac{1}{4d(n)} \int_{B_r} |\nabla u|^2 \leq \int_{B_{2r}\setminus B_r} |\nabla u|^2 \]  

(20)

as required. \[ \blacksquare \]