I. Consider the energy-momentum tensor corresponding to the linear wave equation: $T_{\mu \nu} \overset{\text{def}}{=} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m_{\mu \nu} (m^{-1})^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi$, and assume that $|\nabla_{t,x} \phi| \overset{\text{def}}{=} \sqrt{(\partial_i \phi)^2 + \sum_{i=1}^{n} (\partial_i \phi)^2} \neq 0$. Here, $(m^{-1})^{\mu \nu} = \text{diag}(-1, 1, 1, \cdots, 1)$ is the standard Minkowski metric on $\mathbb{R}^{1+n}$. Let $X,Y$ be future-directed timelike vectors (i.e., $m(X,X) < 0, m(Y,Y) < 0, X^0 > 0$, and $Y^0 > 0$). Show that

\begin{equation}
T(X,Y) \overset{\text{def}}{=} T_{\alpha \beta} X^\alpha Y^\beta > 0.
\end{equation}

**Hint:** First show that if $L$ and $\bar{L}$ are any pair of null vectors normalized by $m(L, L) = -2$, then $T(L, L) \geq 0$, $T(\bar{L}, L) \geq 0$, and that at least one of these three must be non-zero. To prove these facts, it might be helpful to supplement the vectors $L$ and $\bar{L}$ with some vectors $e_{(1)}, \cdots, e_{(n-1)}$ in order to form a null frame $\mathcal{N} \overset{\text{def}}{=} \{L, \bar{L}, e_{(1)}, \cdots, e_{(n-1)}\}$; the calculations will be much easier to do relative to the basis $\mathcal{N}$ compared to the standard basis for $\mathbb{R}^{1+n}$. Recall that $\mathcal{N} \overset{\text{def}}{=} \{L, \bar{L}, e_{(1)}, \cdots, e_{(n-1)}\}$ is any basis for $\mathbb{R}^{1+n}$ such that $0 = m(L, L) = m(\bar{L}, \bar{L}) = m(L, e_{(i)}) = m(\bar{L}, e_{(i)})$ for $1 \leq i \leq n-1$, such that $m(L, \bar{L}) = -2$, such that $m(e_{(i)}, e_{(j)}) = 1$ if $i = j$, and such that $m(e_{(i)}, e_{(j)}) = 0$ if $i \neq j$; as we discussed in class, given any null pair $L, \bar{L}$ normalized by $m(L, \bar{L}) = -2$, there exists such a null frame $\mathcal{N}$ containing $L$ and $\bar{L}$. Recall also that $(m^{-1})^{\mu \nu} = -\frac{1}{2} L^\mu L^\nu - \frac{1}{2} \bar{L}^\mu \bar{L}^\nu + \bar{m}^{\mu \nu}$, where $\bar{m}^{\mu \nu}$ is positive definite on $\text{span}\{e_{(1)}, \cdots, e_{(n-1)}\}$, $\bar{m}^{\mu \nu}$ vanishes on $\text{span}\{L, \bar{L}\}$, and $\bar{m}(L, e_{(i)}) = \bar{m}(\bar{L}, e_{(i)}) = 0$ for $1 \leq i \leq n-1$.

To tackle the case of general $X$ and $Y$, use Problem V from last week.

**Remark 0.0.1.** Inequality (0.0.1) also holds if $X,Y$ are past-directed timelike vectors (i.e., $m(X,X) < 0, m(Y,Y) < 0, X^0 < 0$, and $Y^0 < 0$).

II. Consider the Morawetz vectorfield $\bar{K}^\mu$ on $R^{1+3}$ defined by

\begin{align}
\bar{K}^0 &= 1 + t^2 + (x^1)^2 + (x^2)^2 + (x^3)^2, \\
\bar{K}^j &= 2txt^j, \quad (j = 1, 2, 3).
\end{align}

(a) Show that $\bar{K}$ is future-directed and timelike. Above, $(t, x^1, x^2, x^3)$ are the standard coordinates on $R^{1+3}$.

(b) Show that

\begin{equation}
\partial_\mu \bar{K}^\mu + \partial_\nu \bar{K}^\nu = 4t m_{\mu \nu}, \quad (\mu, \nu = 0, 1, 2, 3),
\end{equation}

where $m_{\mu \nu}$ denotes the Minkowski metric.
Remark 0.0.2. \( \mathbf{K} \) is said to be a conformal Killing field of the Minkowski metric because the right-hand side of (0.0.4) is proportional to \( m_{\mu\nu} \).

c) Show that

\[ m_{\mu\nu}T^{\mu\nu} = 0, \]

where \( T^{\mu\nu} \equiv (m^{-1})^\mu_\alpha (m^{-1})^\nu_\beta T_{\alpha\beta} \) is the energy-momentum tensor from Problem I with its indices raised. Note that the formula (0.0.5) only holds in \( 1 + 3 \) spacetime dimensions.

d) Show that

\[ \partial_\mu(\mathbf{K})^\mu = 0 \]

whenever \( \phi \) is a solution to the linear wave equation

\[ (m^{-1})^\mu_\nu \partial_\mu \phi \partial_\nu \phi = 0, \]

where

\[ \mathbf{K}^\mu = \frac{1}{2} \left\{ \left[ 1 + (r + t)^2 \right] (\nabla_L \phi)^2 + \left[ 1 + (r - t)^2 \right] (\nabla_{\overline{L}} \phi)^2 + 2[1 + t^2 + r^2] \right\} m^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \]

Above, \( (m^{-1})^\mu_\nu = -\frac{1}{2} L^\mu L^\nu - \frac{1}{2} L^\mu \overline{L}^\nu + \partial^{\mu\nu} \) is the standard null decomposition of \( (m^{-1})^\mu_\nu \) from class. In particular, \( L^\mu = (1, \frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}) \), \( \overline{L}^\mu = (1, -\frac{x_1}{r}, -\frac{x_2}{r}, -\frac{x_3}{r}) \), \( \nabla_L \phi = \partial_t \phi + \partial_r \phi \), \( \nabla_{\overline{L}} \phi = \partial_t \phi - \partial_r \phi \), and \( \partial^{\mu\nu} \partial_\phi \partial_\nu \phi \) is the square of the Euclidean norm of the angular derivatives of \( \phi \). Here, \( r \equiv \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \) denotes the standard spherical coordinate on \( \mathbb{R}^3 \), and \( \partial_r \) denotes the standard radial derivative.

Hint: The following expansions in terms of \( L \) and \( \overline{L} \) may be very helpful:

\[ \mathbf{K}^\mu = \frac{1}{2} \left\{ 1 + (r^2 + t^2) L^\mu + 1 + (r^2 - t^2) \overline{L}^\mu \right\}, \]

\[ (1, 0, 0, 0) = \frac{1}{2} (L^\mu + \overline{L}^\mu), \]

\[ (\mathbf{K})^\mu = T(\mathbf{K}, \frac{1}{2} (L + \overline{L})) \]

\[ = \frac{1}{4} \left\{ 1 + (r + t)^2 T(L, L) + 1 + (r - t)^2 T(\overline{L}, \overline{L}) + [1 + (r + t)^2] + [1 + (r - t)^2] \right\} T(L, \overline{L}). \]

f) Finally, with the help of the vectorfield \((\mathbf{K})^\mu \), apply the divergence theorem on an appropriately chosen spacetime region and use the previous results to derive the following conservation law for smooth solutions to the linear wave equation \((m^{-1})^\mu_\nu \partial_\mu \phi \partial_\nu \phi = 0\)
\begin{align}
\int_{\mathbb{R}^4} \frac{1}{4} \left\{ [1 + (t + r)^2] (\nabla L \phi(t, x))^2 + [1 + (t - r)^2] (\nabla L \phi(t, x))^2 + 2 [1 + t^2 + r^2] \right. \\
\left. f^{\mu \nu} \partial_\mu \phi(t, x) \partial_\nu \phi(t, x) \right\} d^3 x \\
= \int_{\mathbb{R}^4} \frac{1}{4} \left\{ [1 + r^2] (\nabla L \phi(0, x))^2 + [1 + r^2] (\nabla L \phi(0, x))^2 + 2 [1 + r^2] \right. \\
\left. f^{\mu \nu} \partial_\mu \phi(0, x) \partial_\nu \phi(0, x) \right\} d^3 x.
\end{align}

For simplicity, at each fixed $t$, you may assume that there exists an $R > 0$ such that $\phi(t, x)$ vanishes whenever $|x| \geq R$.

**Remark 0.0.3.** Note that the right-hand side of (0.0.11) can be computed in terms of the initial data alone. Note also that the different null derivatives of $\phi$ appearing on the left-hand side of (0.0.11) carry different weights. In particular, $\nabla L \phi$ and the angular derivatives of $\phi$ have larger weights than $\nabla L \phi$. These larger weights are strongly connected to the following fact, whose full proof requires additional methods going beyond this course: $\nabla L \phi$ and the angular derivatives of $\phi$ decay faster in $t$ compared to $\nabla L \phi$. 
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