

MATH 18.152 - MIDTERM EXAM

18.152 Introduction to PDEs, Fall 2011

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Midterm Exam, Thursday, October 27

Answer questions I - V below. Each question is worth 20 points, for a total of 100. Partial credit may be awarded, but only if you show all of your work and it is in a logical order. In order to receive credit, whenever you make use of a theorem/proposition, make sure that you state it by name. Also, clearly state the hypotheses that are needed to the apply theorem/ proposition, and explain why the hypotheses are satisfied.

I. Consider the initial/boundary-value problem for the “backwards heat equation” (note carefully the minus sign) in one spatial dimension:

$$(1) \quad \partial_t u = -\partial_x^2 u, \quad (t, x) \in [0, 1] \times [0, 1].$$

a) (8 pts.) Use the separation of variables method to find all solutions to the PDE (1) of the form $u(t, x) = v(t)w(x)$ that satisfy the boundary conditions $u(t, 0) = 0$ and $u(t, 1) = 0$ for $t \in [0, 1]$.

b) (12 pts.) Let $f(x)$ be a *smooth* (i.e., infinitely differentiable) function on $[0, 1]$ with $f(0) = f(1) = 0$. Discuss the behavior of the solution $u(t, x)$ to the following initial/boundary-value problem:

$$(2a) \quad \partial_t u = -\partial_x^2 u, \quad (t, x) \in [0, 1] \times [0, 1],$$

$$(2b) \quad u(0, x) = f(x), \quad x \in [0, 1],$$

$$(2c) \quad u(t, 0) = 0, \quad u(t, 1) = 0, \quad t \in [0, 1].$$

In particular, under the assumption $\max_{x \in [0, 1]} |f(x)| \leq \epsilon$, where ϵ is a small positive number, explain what conclusions can be reached about the “size” of the solution at time $t = 1$. The term “size” is defined here to be $\max_{x \in [0, 1]} |u(t, x)|$. Compare/contrast this behavior with the case $f = 0$ (in which case the solution is $u(t, x) = 0$ for $(t, x) \in [0, 1] \times [0, 1]$). In addition, compare/contrast the behavior with the behavior of the corresponding solutions that arise when equation (2a) is replaced with the ordinary heat equation $\partial_t u = \partial_x^2 u$. Finally, explain how your previous remarks are connected to the question of the well-posedness of the problem (2a) - (2c).

II. Let $u \in C^\infty(\mathbb{R}^3)$ be a harmonic function on \mathbb{R}^3 :

$$\Delta u(x) = 0, \quad x \in \mathbb{R}^3.$$

Assume that $|u(x)| \leq \sqrt{|x|}$ holds for all x , where $x \stackrel{\text{def}}{=} (x^1, x^2, x^3)$, and $|x| \stackrel{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Show that $u(x) = 0$ for all $x \in \mathbb{R}^3$.

III. Let $u(t, x) \in C^{1,2}([0, 2] \times [0, 1])$ be a solution to the following initial/boundary-value problem:

$$(3) \quad \begin{aligned} \partial_t u - \partial_x^2 u &= -u, & (t, x) \in [0, 2] \times [0, 1], \\ u(0, x) &= f(x), & x \in [0, 1], \\ u(t, 0) &= g(t), & u(t, 1) = h(t), & t \in [0, 2]. \end{aligned}$$

Assume that $f(x) \leq 0$ for $x \in [0, 1]$ and that $g(t) \leq 0, h(t) \leq 0$ for $t \in [0, 2]$. Prove that $u(t, x) \leq 0$ holds for all $(t, x) \in [0, 2] \times [0, 1]$.

Hint: One possible approach would be to revisit the **proof** of the maximum principle for the heat equation.

IV. Let $u \in C^{1,2}([0, \infty) \times [0, 1])$ be a solution to the problem

$$(4) \quad \begin{aligned} \partial_t u - \partial_x^2 u &= 0, & (t, x) \in [0, \infty) \times [0, 1], \\ u(0, x) &= f(x), & x \in [0, 1], \\ \partial_x u(t, 0) &= 0, & \partial_x u(t, 1) = 0, & t \in [0, \infty). \end{aligned}$$

Define the *thermal energy* \mathcal{T} of u at time t by

$$(5) \quad \mathcal{T}(t) \stackrel{\text{def}}{=} \int_{[0,1]} u(t, x) dx.$$

a) (6 pts.) Show that under the above assumptions, $\mathcal{T}(t)$ is constant in time: $\mathcal{T}(t) = \mathcal{T}(0)$ for all $t \geq 0$.

b) (2 pts.) Using your physical and mathematical intuition, make an educated guess about what happens to $u(t, x)$ as $t \rightarrow \infty$. In particular, relate your guess to an integral of the data $f(x)$.

c) (12 pts.) Rigorously prove that your guess from part **b)** is correct.

Remark 0.0.1. There are several different ways to approach the proof of **c)**, and the different approaches will lead to slightly different, but related conclusions. In particular, your answer could be stated using any one of several different norms. You will receive full credit if you correctly implement any of the approaches.

V. Assume that $\mathfrak{F}(t, x) \in C^2([0, \infty) \times \mathbb{R})$, that $f(x) \in C^2(\mathbb{R}) \cap L^2(\mathbb{R})$, and that $g(x) \in C^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Let $u(t, x) \in C^2([0, \infty) \times \mathbb{R})$ be the solution to the following global Cauchy problem for an inhomogeneous wave equation in one spatial dimension:

$$\begin{aligned} -\partial_t^2 u(t, x) + \partial_x^2 u(t, x) &= \mathfrak{F}(t, x), & (t, x) \in [0, \infty) \times \mathbb{R}, \\ u(0, x) &= f(x), & \partial_t u(0, x) = g(x). \end{aligned}$$

Assume that at each fixed t ,

$$\|\mathfrak{F}(t, \cdot)\|_{L^2} \leq \frac{1}{1+t^2},$$

where $\|\mathfrak{F}(t, \cdot)\|_{L^2} \stackrel{\text{def}}{=} \left(\int_{\mathbb{R}} |\mathfrak{F}(t, x)|^2 dx \right)^{1/2}$. Also assume for simplicity that at each fixed t , there exists a positive number $R(t)$ such that $u(t, x) = 0$ whenever $|x| \geq R(t)$.

Define the square of the solution's energy E at time t by

$$E^2(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} (\partial_t u(t, x))^2 + (\partial_x u(t, x))^2 dx.$$

a) (8 pts.) Show that

$$(6) \quad \frac{d}{dt} E^2(t) = -2 \int_{\mathbb{R}} \mathfrak{F}(t, x) \partial_t u(t, x) dx.$$

b) (12 pts.) Then show that the following energy inequality holds for all $t \geq 0$:

$$E(t) \leq E(0) + C,$$

where $C > 0$ is a constant that you do *not* have to explicitly evaluate.

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