Problem set 2: Due September 28

From Notes: Problems 6, 11, 12, 13, 14.

Problem 1  Show that the smallest $\sigma$-algebra containing the sets

$$(a, \infty) \subset [-\infty, \infty]$$

for all $a \in \mathbb{R}$, is what is called above the `Borel' $\sigma$-algebra on \((X, \mathcal{M}, \mu)\).

Problem 2  Let \((X, \mathcal{M}, \mu)\) be any measure space (so \(\mu\) is a measure on the $\sigma$-algebra $\mathcal{M}$ of subsets of $X$). Show that the set of equivalence classes of $\mu$-integrable functions on $X$, with the equivalence relation

$$f_1 \equiv f_2 \iff \mu(\{x \in X; f_1(x) \neq f_2(x)\}) = 0.$$

is a normed linear space with the usual linear structure and the norm given by

$$\|f\| = \int_X |f| d\mu.$$

Problem 3  Let \((X, \mathcal{M})\) be a set with a $\sigma$-algebra. Let $\mu : \mathcal{M} \rightarrow \mathbb{R}$ be a finite measure in the sense that $\mu(\emptyset) = 0$ and for any \(\{E_i\}_{i=1}^{\infty} \subset \mathcal{M} \) with $E_i \cap E_j = \emptyset$ for $i \neq j$,

$$\mu \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i) \quad (1)$$

with the series on the right always absolutely convergent (i.e., this is part of the requirement on $\mu$). Define

$$|\mu| (E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \quad (2)$$

for $E \in \mathcal{M}$, with the supremum over all measurable decompositions $E = \bigcup_{i=1}^{\infty} E_i$ with $E_i$ disjoint. Show that $|\mu|$ is a finite, positive measure.
Hint 1. You must show that
\[ |\mu| (E) = \sum_{i=1}^{\infty} |\mu| (A_i) \quad \text{if} \quad \bigcup_{i} A_i = E, \quad A_i \in \mathcal{M} \]
being disjoint. Observe that if \( A_j = \bigcup_{i} A_{ji} \) is a measurable decomposition of \( A_j \) then together the \( A_{ji} \) give a decomposition of \( E \). Similarly, if \( A_{ji} = A_j \cap E_i \) is any such decomposition of \( E \) then gives such a decomposition of \( A_j \).

**Problem 4 (Hahn Decomposition)**

With assumptions as in Problem 3:

1. Show that \( \mu_+ = \frac{1}{2}(|\mu| + \mu) \) and \( \mu_- = \frac{1}{2}(|\mu| - \mu) \) are positive measures, \( \mu = \mu_+ - \mu_- \). Conclude that the definition of a measure in the notes based on (4.17) is the same as that in Problem 3.

2. Show that \( \mu_\pm \) so constructed are orthogonal in the sense that there is a set \( E \in \mathcal{M} \) \( \mu_- (E) = 0, \mu_+ (X \setminus E) = 0 \).

**Hint.** Use the definition of \( |\mu| \) to show that for any \( F \in \mathcal{M} \) and any \( \epsilon > 0 \) there is a subset \( F' \in \mathcal{M}, F' \subseteq F \) such that \( \mu_+ (F') \geq \mu_+ (F) - \epsilon \) and \( \mu_-(F') \leq \epsilon \) for some \( \delta > 0 \) and \( \epsilon = 2^{-n} \delta \).

Given apply this result repeatedly (say with to find a decreasing sequence of sets \( \mu_+ (F_n) \geq \mu_+ (F_{n-1}) - 2^{-n} \delta \) \( \mu_- (F_n) \leq 2^{-n} \delta \).

Now let be chosen this way with \( \delta = 1/m \).

Show that \( E = \bigcup_m G_m \) is as required.

**Problem 5**

Now suppose that \( \mu \) is a finite, positive Radon measure on a locally compact metric space \( X \) (meaning a finite positive Borel measure outer regular on Borel sets and inner regular on open sets). Show that \( \mu \) is inner regular on all Borel sets and hence, given \( \epsilon > 0 \) and
\( E \in B(X) \) with \( K \) compact and \( U \) open such that
\[
\mu(K) \geq \mu(E) - \epsilon \quad \mu(E) \geq \mu(U) - \epsilon
\]

**Hint.** First take \( U \) open, then use its inner regularity to find \( K \) with 
\[ K' \Subset U \]
and
\[ \mu(K') \geq \mu(U) - \epsilon/2 \quad \mu(E \setminus K') \quad V \supset K' \setminus E \]

How big is \( K' \)? Find \( K = K' \setminus V \).