18. Solutions to (some of) the problems

Solution 18.1 (To Problem 10). (by Matjaž Konvalinka).

Since the topology on $\mathbb{N}$, inherited from $\mathbb{R}$, is discrete, a set is compact if and only if it is finite. If a sequence $\{x_n\}$ (i.e. a function $\mathbb{N} \to \mathbb{C}$) is in $C_0(\mathbb{N})$ if and only if for any $\epsilon > 0$ there exists a compact (hence finite) set $F_\epsilon$ so that $|x_n| < \epsilon$ for any $n$ not in $F_\epsilon$. We can assume that $F_\epsilon = \{1, \ldots, n_\epsilon\}$, which gives us the condition that $\{x_n\}$ is in $C_0(\mathbb{N})$ if and only if it converges to 0. We denote this space by $c_0$, and the supremum norm by $\| \cdot \|_0$. A sequence $\{x_n\}$ will be abbreviated to $x$.

Let $l^1$ denote the space of (real or complex) sequences $x$ with a finite 1-norm

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|.$$  

We can define pointwise summation and multiplication with scalars, and $(l^1, \| \cdot \|_1)$ is a normed (in fact Banach) space. Because the functional

$$y \mapsto \sum_{n=1}^{\infty} x_n y_n$$

is linear and bounded ($|\sum_{n=1}^{\infty} x_n y_n| \leq \sum_{n=1}^{\infty} |x_n||y_n| \leq \|x\|_0 \|y\|_1$) by $\|x\|_0$, the mapping

$$\Phi: l^1 \longrightarrow c_0^*$$

defined by

$$x \mapsto \left( y \mapsto \sum_{n=1}^{\infty} x_n y_n \right)$$

is a (linear) well-defined mapping with norm at most 1. In fact, $\Phi$ is an isometry because if $|x_j| = \|x\|_0$ then $|\Phi(x)(e_j)| = 1$ where $e_j$ is the $j$-th unit vector. We claim that $\Phi$ is also surjective (and hence an isometric isomorphism). If $\varphi$ is a functional on $c_0$ let us denote $\varphi(e_j)$ by $x_j$. Then $\Phi(x)(y) = \sum_{n=1}^{\infty} \varphi(e_n)y_n = \sum_{n=1}^{\infty} \varphi(y_n e_n) = \varphi(y)$ (the last equality holds because $\sum_{n=1}^{\infty} y_n e_n$ converges to $y$ in $c_0$ and $\varphi$ is continuous with respect to the topology in $c_0$), so $\Phi(x) = \varphi$.

Solution 18.2 (To Problem 29). (Matjaž Konvalinka) Since

$$D_x H(\varphi) = H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x) \, dx =$$

$$i \int_{0}^{\infty} \varphi'(x) \, dx = i(0 - \varphi(0)) = -i \delta(\varphi),$$

we get $D_x H = C \delta$ for $C = -i$.  

Solution 18.3 (To Problem 40). (Matjaž Konvalinka) Let us prove this in the case where \( n = 1 \). Define (for \( b \neq 0 \))

\[
U(x) = u(b) - u(x) - (b - x)u'(x) - \ldots - \frac{(b - x)^{k-1}}{(k - 1)!} u^{(k-1)}(x);
\]

then

\[
U'(x) = -\frac{(b - x)^{k-1}}{(k - 1)!} u^{(k)}(x).
\]

For the continuously differentiable function \( V(x) = U(x) - (1 - x/b)^k U(0) \) we have \( V(0) = V(b) = 0 \), so by Rolle’s theorem there exists \( \zeta \) between 0 and \( b \) with

\[
V'(\zeta) = U'(\zeta) + \frac{k(b - \zeta)^{k-1}}{b^k} U(0) = 0
\]

Then

\[
U(0) = -\frac{b^k}{k(b - \zeta)^{k-1}} U'(\zeta),
\]

\[
 u(b) = u(0) + u'(0)b + \ldots + \frac{u^{(k-1)}(0)}{(k - 1)!} b^{k-1} + \frac{u^{(k)}(\zeta)}{k!} b^k.
\]

The required decomposition is \( u(x) = p(x) + v(x) \) for

\[
p(x) = u(0) + u'(0)x + \frac{u''(0)}{2} x^2 + \ldots + \frac{u^{(k-1)}(0)}{(k - 1)!} x^{k-1} + \frac{u^{(k)}(0)}{k!} x^k,
\]

\[
v(x) = u(x) - p(x) = \frac{u^{(k)}(\zeta) - u^{(k)}(0)}{k!} x^k
\]

for \( \zeta \) between 0 and \( x \), and since \( u^{(k)} \) is continuous, \( (u(x) - p(x))/x^k \) tends to 0 as \( x \) tends to 0.

The proof for general \( n \) is not much more difficult. Define the function \( w_x : I \to \mathbb{R} \) by \( w_x(t) = u(tx) \). Then \( w_x \) is \( k \)-times continuously differentiable,

\[
w'_x(t) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(tx)x_i,
\]

\[
w''_x(t) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(tx)x_ix_j,
\]

\[
w^{(l)}_x(t) = \sum_{l_1+l_2+\ldots+l_i=l} \frac{l!}{l_1! l_2! \ldots l_i!} \frac{\partial^l u}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_l}}(tx)x_{i_1}^l x_{i_2}^l \ldots x_{i_l}^l
\]
so by above \( u(x) = w_x(1) \) is the sum of some polynomial \( p \) (of degree \( k \)), and we have
\[
\frac{u(x) - p(x)}{|x|^k} = \frac{v_x(1)}{|x|^k} = \frac{w_x^{(k)}(\xi) - w_x^{(k)}(0)}{k!|x|^k},
\]
so it is bounded by a positive combination of terms of the form
\[
\left| \frac{\partial^l u}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_i^{l_i}}(\xi) - \frac{\partial^l u}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_i^{l_i}}(0) \right|
\]
with \( l_1 + \ldots + l_i = k \) and \( 0 < \zeta < 1 \). This tends to zero as \( x \to 0 \) because the derivative is continuous.

**Solution 18.4** (Solution to Problem 41). (Matjž Konvalinka) Obviously the map \( C_0(\mathbb{B}^n) \to C(\mathbb{B}^n) \) is injective (since it is just the inclusion map), and \( f \in C(\mathbb{B}^n) \) is in \( C_0(\mathbb{B}^n) \) if and only if it is zero on \( \partial \mathbb{B}^n \), i.e. if and only if \( f|_{\mathbb{S}^{n-1}} = 0 \). It remains to prove that any map \( g \) on \( \mathbb{S}^{n-1} \) is the restriction of a continuous function on \( \mathbb{B}^n \). This is clear since
\[
f(x) = \begin{cases} |x|g(x/|x|) & x \neq 0 \\ 0 & x = 0 \end{cases}
\]
is well-defined, coincides with \( f \) on \( \mathbb{S}^{n-1} \), and is continuous: if \( M \) is the maximum of \( |g| \) on \( \mathbb{S}^{n-1} \), and \( \epsilon > 0 \) is given, then \( |f(x)| < \epsilon \) for \( |x| < \epsilon/M \).

**Solution 18.5.** (partly Matjaž Konvalinka)
For any \( \varphi \in S(\mathbb{R}) \) we have
\[
|\int_{-\infty}^{\infty} \varphi(x) dx| \leq \int_{-\infty}^{\infty} |\varphi(x)| dx \leq C \sup((1+|x|^2)|\varphi(x)|) \int_{-\infty}^{\infty} (1+|x|^2)^{-1} dx 
\]
\[
\leq C \sup((1+|x|^2)|\varphi(x)|).
\]
Thus \( S(\mathbb{R}) \ni \varphi \mapsto \int_{\mathbb{R}} \varphi dx \) is continuous.

Now, choose \( \phi \in C_c^\infty(\mathbb{R}) \) with \( \int_{\mathbb{R}} \phi(x) dx = 1 \). Then, for \( \psi \in S(\mathbb{R}) \), set
\[
\psi(x) = \int_{-\infty}^{x} (\psi(t) - c(\psi)\phi(t)) \, dt, \quad c(\psi) = \int_{-\infty}^{\infty} \psi(s) \, ds.
\]
Note that the assumption on \( \phi \) means that
\[
A\psi(x) = -\int_{x}^{\infty} (\psi(t) - c(\psi)\phi(t)) \, dt
\]
Clearly \( A\psi \) is smooth, and in fact it is a Schwartz function since
\[
\frac{d}{dx}(A\psi(x)) = \psi(x) - c\phi(x) \in S(\mathbb{R})
\]
so it suffices to show that $x^k A\psi$ is bounded for any $k$ as $|x| \to \pm \infty$. Since $\psi(t) - c\phi(t) \leq C_k t^{-k-1}$ in $t \geq 1$ it follows from (18.2) that

$$|x^k A\psi(x)| \leq C x^k \int_x^{\infty} t^{-k-1} dt \leq C', \ k > 1, \ in \ x > 1.$$ 

A similar estimate as $x \to -\infty$ follows from (18.1). Now, $A$ is clearly linear, and it follows from the estimates above, including that on the integral, that for any $k$ there exists $C$ and $j$ such that

$$\sup_{\alpha, \beta \leq k} |x^\alpha D^\beta A\psi| \leq C \sum_{\alpha', \beta' \leq j} \sup_{x \in \mathbb{R}} |x^{\alpha'} D^{\beta'} \psi|.$$ 

Finally then, given $u \in S'(\mathbb{R})$ define $v(\psi) = -u(A\psi)$. From the continuity of $A$, $v \in S(\mathbb{R})$ and from the definition of $A$, $A(\psi') = \psi$.

Thus

$$dv/dx(\psi) = v(-\psi') = u(A\psi') = u(\psi) \implies \frac{dv}{dx} = u.$$ 

**Solution 18.6.** We have to prove that $\langle \xi \rangle^{m+m'} \hat{u} \in L_2(\mathbb{R}^n)$, in other words, that

$$\int_{\mathbb{R}^n} \langle \xi \rangle^{2(m+m')} |\hat{u}|^2 d\xi < \infty.$$ 

But that is true since

$$\int_{\mathbb{R}^n} \langle \xi \rangle^{2(m+m')} |\hat{u}|^2 d\xi = \int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} (1 + \xi_1^2 + \ldots + \xi_n^2)^m |\hat{u}|^2 d\xi =$$

$$\int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} \left( \sum_{|\alpha| \leq m} C_\alpha \xi^{2\alpha} \right) |\hat{u}|^2 d\xi = \sum_{|\alpha| \leq m} C_\alpha \left( \int_{\mathbb{R}^n} \langle \xi \rangle^{2m'} \xi^{2\alpha} |\hat{u}|^2 d\xi \right)$$

and since $\langle \xi \rangle^{m'} \xi^\alpha \hat{u} = \langle \xi \rangle^{m'} \hat{D^\alpha u}$ is in $L^2(\mathbb{R}^n)$ (note that $u \in H^m(\mathbb{R}^n)$ follows from $D^\alpha u \in H^m(\mathbb{R}^n)$, $|\alpha| \leq m$). The converse is also true since $C_\alpha$ in the formula above are strictly positive.

**Solution 18.7.** Take $v \in L^2(\mathbb{R}^n)$, and define subsets of $\mathbb{R}^n$ by

$$E_0 = \{ x : |x| \leq 1 \},$$

$$E_i = \{ x : |x| \geq 1, |x_i| = \max_j |x_j| \}.$$ 

Then obviously we have $1 = \sum_{i=0}^n \chi_{E_i}$ a.e., and $v = \sum_{j=0}^n v_j$ for $v_j = \chi_{E_j} v$. Then $\langle x \rangle$ is bounded by $\sqrt{2}$ on $E_0$, and $\langle x \rangle v_0 \in L^2(\mathbb{R}^n)$; and on $E_j$, $1 \leq j \leq n$, we have

$$\frac{\langle x \rangle}{|x_j|} \leq \frac{(1 + n|x_j|^2)^{1/2}}{|x_j|} = (n + 1/|x_j|^2)^{1/2} \leq (2n)^{1/2},$$

$$\langle x \rangle \leq \frac{(1 + n|x_j|^2)^{1/2}}{|x_j|} = (n + 1/|x_j|^2)^{1/2} \leq (2n)^{1/2},$$

and since $\langle \xi \rangle^{m'} \xi^\alpha \hat{u} = \langle \xi \rangle^{m'} \hat{D^\alpha u}$ is in $L^2(\mathbb{R}^n)$ (note that $u \in H^m(\mathbb{R}^n)$ follows from $D^\alpha u \in H^m(\mathbb{R}^n)$, $|\alpha| \leq m$). The converse is also true since $C_\alpha$ in the formula above are strictly positive.
so \( \langle x \rangle v_j = x_j w_j \) for \( w_j \in L^2(\mathbb{R}^n) \). But that means that \( \langle x \rangle v = w_0 + \sum_{j=1}^{n} x_j w_j \) for \( w_j \in L^2(\mathbb{R}^n) \).

If \( u \) is in \( L^2(\mathbb{R}^n) \) then \( \hat{u} \in L^2(\mathbb{R}^n) \), and so there exist \( w_0, \ldots, w_n \in L^2(\mathbb{R}^n) \) so that

\[
\langle \xi \rangle \hat{u} = w_0 + \sum_{j=1}^{n} \xi_j w_j,
\]

in other words

\[
\hat{u} = \hat{u}_0 + \sum_{j=1}^{n} \xi_j \hat{u}_j
\]

where \( \langle \xi \rangle \hat{u}_j \in L^2(\mathbb{R}^n) \). Hence

\[
u = u_0 + \sum_{j=1}^{n} D_j u_j
\]

where \( u_j \in H^1(\mathbb{R}^n) \).

**Solution 18.8.** Since

\[
D_x H(\varphi) = H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x) \, dx = i \int_{0}^{\infty} \varphi'(x) \, dx = i(0 - \varphi(0)) = -i \delta(\varphi),
\]

we get \( D_x H = C\delta \) for \( C = -i \).

**Solution 18.9.** It is equivalent to ask when \( \langle \xi \rangle^m \hat{\delta}_0 \) is in \( L^2(\mathbb{R}^n) \). Since

\[
\hat{\delta}_0(\psi) = \delta_0(\hat{\psi}) = \hat{\psi}(0) = \int_{\mathbb{R}^n} \psi(x) \, dx = 1(\psi),
\]

this is equivalent to finding \( m \) such that \( \langle \xi \rangle^{2m} \) has a finite integral over \( \mathbb{R}^n \). One option is to write \( \langle \xi \rangle = (1 + r^2)^{1/2} \) in spherical coordinates, and to recall that the Jacobian of spherical coordinates in \( n \) dimensions has the form \( r^{n-1} \Psi(\varphi_1, \ldots, \varphi_{n-1}) \), and so \( \langle \xi \rangle^{2m} \) is integrable if and only if

\[
\int_{0}^{\infty} \frac{r^{n-1}}{(1 + r^2)^m} \, dr
\]

converges. It is obvious that this is true if and only if \( n - 1 - 2m < -1 \), ie. if and only if \( m > n/2 \).

**Solution 18.10** (Solution to Problem31). We know that \( \delta \in H^m(\mathbb{R}^n) \) for any \( m < -n/1 \). Thus is just because \( \langle \xi \rangle^p \in L^2(\mathbb{R}^n) \) when \( p < -n/2 \).

Now, divide \( \mathbb{R}^n \) into \( n + 1 \) regions, as above, being \( A_0 = \{ \xi \in 1 \leq |\xi| \} \) and

\[
A_i = \{ \xi \in 1 \leq |\xi|, |\xi| \geq 1 \}.
\]

Let \( v_0 \) have Fourier transform \( \chi_{A_0} \) and for \( i = 1, \ldots, n \), \( v_i \in \mathcal{S}(\mathbb{R}^n) \) have Fourier transforms \( \xi_i^{-n-1} \chi_{A_i} \).

Since \( |\xi_i| > c \langle \xi \rangle \) on the support of \( \hat{v}_i \), for each \( i = 1, \ldots, n \), each term
is in $H^m$ for any $m < 1 + n/2$ so, by the Sobolev embedding theorem, each $v_i \in C^0(\mathbb{R}^n)$ and

$$1 = \hat{v}_0 \sum_{i=1}^{n} \xi_i^{n+1} \hat{v}_i \implies \delta = v_0 + \sum_{i} D_i^{n+1} v_i. \quad (18.4)$$

How to see that this cannot be done with $n$ or less derivatives? For the moment I do not have a proof of this, although I believe it is true. Notice that we are actually proving that $\delta$ can be written

$$\delta = \sum_{|\alpha| \leq n+1} D^\alpha u_\alpha, \ u_\alpha \in H^{n/2}(\mathbb{R}^n). \quad (18.5)$$

This cannot be improved to $n$ from $n + 1$ since this would mean that $\delta \in H^{-n/2}(\mathbb{R}^n)$, which it isn’t. However, what I am asking is a little more subtle than this.