16. Spectral theorem

For a bounded operator $T$ on a Hilbert space we define the spectrum as the set

\[(16.1) \quad \text{spec}(T) = \{ z \in \mathbb{C}; T - z \text{Id} \text{ is not invertible} \}.\]

**Proposition 16.1.** For any bounded linear operator on a Hilbert space $\text{spec}(T) \subset \mathbb{C}$ is a compact subset of $\{ |z| \leq \|T\| \}$.

**Proof.** We show that the set $\mathbb{C} \setminus \text{spec}(T)$ (generally called the resolvent set of $T$) is open and contains the complement of a sufficiently large ball. This is based on the convergence of the Neumann series. Namely if $T$ is bounded and $\|T\| < 1$ then

\[(16.2) \quad (\text{Id} - T)^{-1} = \sum_{j=0}^{\infty} T^j \]

converges to a bounded operator which is a two-sided inverse of $\text{Id} - T$. Indeed, $\|T^j\| \leq \|T\|^j$ so the series is convergent and composing with $\text{Id} - T$ on either side gives a telescoping series reducing to the identity.

Applying this result, we first see that

\[(16.3) \quad (T - z) = -z(\text{Id} - T/z) \]

is invertible if $|z| > \|T\|$. Similarly, if $(T - z_0)^{-1}$ exists for some $z_0 \in \mathbb{C}$ then

\[(16.4) \quad (T - z) = (T - z_0) - (z - z_0) = (T - z_0)^{-1}(\text{Id} - (z - z_0)(T - z_0)^{-1}) \]

exists for $|z - z_0|\|(T - z_0)^{-1}\| < 1$. \hfill \qed

In general it is rather difficult to precisely locate $\text{spec}(T)$.

However for a bounded self-adjoint operator it is easier. One sign of this is the the norm of the operator has an alternative, simple, characterization. Namely

\[(16.5) \quad \text{if } A^* = A \text{ then } \sup_{\|\phi\|=1} \langle A\phi, \phi \rangle = \|A\|. \]

If $a$ is this supremum, then clearly $a \leq \|A\|$. To see the converse, choose any $\phi, \psi \in H$ with norm 1 and then replace $\psi$ by $e^{i\theta}\psi$ with $\theta$ chosen so that $\langle A\phi, \psi \rangle$ is real. Then use the polarization identity to write

\[(16.6) \quad 4\langle A\phi, \psi \rangle = \langle A(\phi + \psi), (\phi + \psi) \rangle - \langle A(\phi - \psi), (\phi - \psi) \rangle + i\langle A(\phi + i\psi), (\phi + i\psi) \rangle - i\langle A(\phi - i\psi), (\phi - i\psi) \rangle. \]

Now, by the assumed reality we may drop the last two terms and see that

\[(16.7) \quad 4|\langle A\phi, \psi \rangle| \leq a(\|\phi + \psi\|^2 + \|\phi - \psi\|^2) = 2a(\|\phi\|^2 + \|\psi\|^2) = 4a. \]
Thus indeed \( \|A\| = \sup_{\|\phi\| = \|\psi\| = 1} |\langle A\phi, \psi \rangle| = a. \)

We can always subtract a real constant from \( A \) so that \( A' = A - t \) satisfies

\[
(16.8) \quad -\inf_{\|\phi\| = 1} \langle A'\phi, \phi \rangle = \sup_{\|\phi\| = 1} \langle A'\phi, \phi \rangle = \|A'\|.
\]

Then, it follows that \( A' + \|A'\| \) is not invertible. Indeed, there exists a sequence \( \phi_n \), with \( \|\phi_n\| = 1 \) such that \( \langle (A' + \|A'\|)\phi_n, \phi_n \rangle \to 0 \). Thus

\[
(16.9) \quad \|(A' - \|A'\|)\phi_n\|^2 = -2\langle A'\phi_n, \phi_n \rangle + \|A'\phi_n\|^2 + \|A'\|^2 \leq -2\langle A'\phi_n, \phi_n \rangle + 2\|A'\|^2 \to 0.
\]

This shows that \( A' + \|A'\| \) cannot be invertible and the same argument works for \( A' - \|A'\| \). For the original operator \( A \) if we set

\[
(16.10) \quad m = \inf_{\|\phi\| = 1} \langle A\phi, \phi \rangle = \sup_{\|\phi\| = 1} \langle A\phi, \phi \rangle
\]

then we conclude that neither \( A - m \text{Id} \) nor \( A - M \text{Id} \) is invertible and \( \|A\| = \max(-m, M) \).

**Proposition 16.2.** If \( A \) is a bounded self-adjoint operator then, with \( m \) and \( M \) defined by (16.10),

\[
(16.11) \quad \{m\} \cup \{M\} \subset \text{spec}(A) \subset [m, M].
\]

**Proof.** We have already shown the first part, that \( m \) and \( M \) are in the spectrum so it remains to show that \( A - z \) is invertible for all \( z \in \mathbb{C} \setminus [m, M] \).

Using the self-adjointness

\[
(16.12) \quad \text{Im} \langle (A - z)\phi, \phi \rangle = -\text{Im} z\|\phi\|^2.
\]

This implies that \( A - z \) is invertible if \( z \in \mathbb{C} \setminus \mathbb{R} \). First it shows that \( (A - z)\phi = 0 \) implies \( \phi = 0 \), so \( A - z \) is injective. Secondly, the range is closed. Indeed, if \( (A - z)\phi_n \to \psi \) then applying (16.12) directly shows that \( \|\phi_n\| \) is bounded and so can be replaced by a weakly convergent subsequence. Applying (16.12) again to \( \phi_n - \phi_m \) shows that the sequence is actually Cauchy, hence converges to \( \phi \) so \( (A - z)\phi = \psi \) is in the range. Finally, the orthocomplement to this range is the null space of \( A^* - \bar{z} \), which is also trivial, so \( A - z \) is an isomorphism and (16.12) also shows that the inverse is bounded, in fact

\[
(16.13) \quad \|(A - z)^{-1}\| \leq \frac{1}{|\text{Im} z|}.
\]

When \( z \in \mathbb{R} \) we can replace \( A \) by \( A' \) satisfying (16.8). Then we have to show that \( A' - z \) is invertible for \( |z| > \|A\| \), but that is shown in the proof of Proposition 16.1. \( \square \)
The basic estimate leading to the spectral theorem is:

**Proposition 16.3.** If $A$ is a bounded self-adjoint operator and $p$ is a real polynomial in one variable,

\[(16.14) \quad p(t) = \sum_{i=0}^{N} c_i t^i, \quad c_N \neq 0,\]

then $p(A) = \sum_{i=0}^{N} c_i A^i$ satisfies

\[(16.15) \quad \|p(A)\| \leq \sup_{t \in [m,M]} |p(t)|.\]

**Proof.** Clearly, $p(A)$ is a bounded self-adjoint operator. If $s \notin p([m, M])$ then $p(A) - s$ is invertible. Indeed, the roots of $p(t) - s$ must lie in $[m,M]$, since otherwise $s \in p([m, M])$. Thus, factorizing $p(s) - t$ we have

\[(16.16) \quad p(t) - s = c_N \prod_{i=1}^{N} (t - t_i(s)), \quad t_i(s) \notin [m, M] \implies (p(A) - s)^{-1} \text{ exists} \]

since $p(A) = c_N \sum_i \prod (t - t_i(s))$ and each of the factors is invertible. Thus $\text{spec}(p(A)) \subset p([m, M])$, which is an interval (or a point), and from Proposition 16.3 we conclude that $\|p(A)\| \leq \sup p([m, M])$ which is (16.15). \qed

Now, reinterpreting (16.15) we have a linear map

\[(16.17) \quad \mathcal{P}(\mathbb{R}) \ni p \longmapsto p(A) \in \mathcal{B}(H)\]

from the real polynomials to the bounded self-adjoint operators which is continuous with respect to the supremum norm on $[m, M]$. Since polynomials are dense in continuous functions on finite intervals, we see that (16.17) extends by continuity to a linear map

\[(16.18) \quad \mathcal{C}([m, M]) \ni f \longmapsto f(A) \in \mathcal{B}(H), \quad \|f(A)\| \leq \|f\|_{[m,M]}, \quad fg(A) = f(A)g(A)\]

where the multiplicativity follows by continuity together with the fact that it is true for polynomials.

Now, consider any two elements $\phi, \psi \in H$. Evaluating $f(A)$ on $\phi$ and pairing with $\psi$ gives a linear map

\[(16.19) \quad \mathcal{C}([m, M]) \ni f \longmapsto \langle f(A)\phi, \psi \rangle \in \mathbb{C}.\]

This is a linear functional on $\mathcal{C}([m, M])$ to which we can apply the Riesz representation theorem and conclude that it is defined by integration.
against a unique Radon measure \( \mu_{\phi, \psi} \):

\[
\langle f(A) \phi, \psi \rangle = \int_{[m,M]} f d\mu_{\phi, \psi}.
\]

The total mass \(|\mu_{\phi, \psi}|\) of this measure is the norm of the functional. Since it is a Borel measure, we can take the integral on \(-\infty, b]\) for any \(b \in \mathbb{R}\) ad, with the uniqueness, this shows that we have a continuous sesquilinear map

\[
P_b(\phi, \psi) : H \times H \ni (\phi, \psi) \mapsto \int_{[m,b]} d\mu_{\phi, \psi} \in \mathbb{R}, \quad |P_b(\phi, \psi)| \leq \|A\| \|\phi\| \|\psi\|.
\]

From the Hilbert space Riesz representation theorem it follows that this sesquilinear form defines, and is determined by, a bounded linear operator

\[
P_b(\phi, \psi) = \langle P_b \phi, \psi \rangle, \quad \|P_b\| \leq \|A\|.
\]

In fact, from the functional calculus (the multiplicativity in (16.18)) we see that

\[
P_b^* = P_b, \quad P_b^2 = P_b, \quad \|P_b\| \leq 1,
\]

so \(P_b\) is a projection.

Thus the spectral theorem gives us an increasing (with \(b\)) family of commuting self-adjoint projections such that \(\mu_{\phi, \psi}([\infty, b]) = \langle P_b \phi, \psi \rangle\) determines the Radon measure for which (16.20) holds. One can go further and think of \(P_b\) itself as determining a measure

\[
\mu([\infty, b]) = P_b
\]

which takes values in the projections on \(H\) and which allows the functions of \(A\) to be written as integrals in the form

\[
f(A) = \int_{[m,M]} f d\mu
\]

of which (16.20) becomes the ‘weak form’. To do so one needs to develop the theory of such measures and the corresponding integrals. This is not so hard but I shall not do it.