3. Measureability of functions

Suppose that \( \mathcal{M} \) is a \( \sigma \)-algebra on a set \( X \)\(^4\) and \( \mathcal{N} \) is a \( \sigma \)-algebra on another set \( Y \). A map \( f : X \to Y \) is said to be \emph{measurable} with respect to these given \( \sigma \)-algebras on \( X \) and \( Y \) if
\[
f^{-1}(E) \in \mathcal{M} \ \forall \ E \in \mathcal{N}.
\]
Notice how similar this is to one of the characterizations of continuity for maps between metric spaces in terms of open sets. Indeed this analogy yields a useful result.

**Lemma 3.1.** If \( G \subset \mathcal{N} \) generates \( \mathcal{N} \), in the sense that
\[
\mathcal{N} = \bigcap \{ \mathcal{N}'; \mathcal{N}' \supset G, \ \mathcal{N}' \text{ a } \sigma\text{-algebra} \}
\]
then \( f : X \to Y \) is measurable iff \( f^{-1}(A) \in \mathcal{M} \) for all \( A \in G \).

**Proof.** The main point to note here is that \( f^{-1} \) as a map on power sets, is very well behaved for any map. That is if \( f : X \to Y \) then \( f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X) \) satisfies:
\[
f^{-1}(E^c) = (f^{-1}(E))^c
\]
\[
f^{-1} \left( \bigcup_{j=1}^{\infty} E_j \right) = \bigcup_{j=1}^{\infty} f^{-1}(E_j)
\]
\[
f^{-1} \left( \bigcap_{j=1}^{\infty} E_j \right) = \bigcap_{j=1}^{\infty} f^{-1}(E_j)
\]
\[
f^{-1}(\phi) = \phi, \ f^{-1}(Y) = X.
\]
Putting these things together one sees that if \( \mathcal{M} \) is any \( \sigma \)-algebra on \( X \) then
\[
f_*(\mathcal{M}) = \{ E \subset Y; f^{-1}(E) \in \mathcal{M} \}
\]
is always a \( \sigma \)-algebra on \( Y \).

In particular if \( f^{-1}(A) \in \mathcal{M} \) for all \( A \in G \subset \mathcal{N} \) then \( f_*(\mathcal{M}) \) is a \( \sigma \)-algebra containing \( G \), hence containing \( \mathcal{N} \) by the generating condition. Thus \( f^{-1}(E) \in \mathcal{M} \) for all \( E \in \mathcal{N} \) so \( f \) is measurable. \( \square \)

**Proposition 3.2.** Any continuous map \( f : X \to Y \) between metric spaces is measurable with respect to the Borel \( \sigma \)-algebras on \( X \) and \( Y \).

\(^4\)Then \( X \), or if you want to be pedantic \((X, \mathcal{M})\), is often said to be a measure space or even a measurable space.
Proof. The continuity of $f$ shows that $f^{-1}(E) \subset X$ is open if $E \subset Y$ is open. By definition, the open sets generate the Borel $\sigma$-algebra on $Y$ so the preceding Lemma shows that $f$ is Borel measurable i.e.,

$$f^{-1}(\mathcal{B}(Y)) \subset \mathcal{B}(X).$$

\[\square\]

We are mainly interested in functions on $X$. If $\mathcal{M}$ is a $\sigma$-algebra on $X$ then $f : X \to \mathbb{R}$ is measurable if it is measurable with respect to the Borel $\sigma$-algebra on $\mathbb{R}$ and $\mathcal{M}$ on $X$. More generally, for an extended function $f : X \to [-\infty, \infty]$ we take as the ‘Borel’ $\sigma$-algebra in $[-\infty, \infty]$ the smallest $\sigma$-algebra containing all open subsets of $\mathbb{R}$ and all sets $(a, \infty]$ and $[-\infty, b)$; in fact it is generated by the sets $(a, \infty]$. (See Problem 6.)

Our main task is to define the integral of a measurable function: we start with simple functions. Observe that the characteristic function of a set

$$\chi_E = \begin{cases} 
1 & x \in E \\
0 & x \notin E 
\end{cases}$$

is measurable if and only if $E \in \mathcal{M}$. More generally a simple function,

$$f = \sum_{i=1}^{N} a_i \chi_{E_i}, \quad a_i \in \mathbb{R}$$

(3.5)

is measurable if the $E_i$ are measurable. The presentation, (3.5), of a simple function is not unique. We can make it so, getting the minimal presentation, by insisting that all the $a_i$ are non-zero and

$$E_i = \{x \in E; f(x) = a_i\}$$

then $f$ in (3.5) is measurable iff all the $E_i$ are measurable.

The Lebesgue integral is based on approximation of functions by simple functions, so it is important to show that this is possible.

Proposition 3.3. For any non-negative $\mu$-measurable extended function $f : X \to [0, \infty]$ there is an increasing sequence $f_n$ of simple measurable functions such that $\lim_{n \to \infty} f_n(x) = f(x)$ for each $x \in X$ and this limit is uniform on any measurable set on which $f$ is finite.

Proof. Folland [1] page 45 has a nice proof. For each integer $n > 0$ and $0 \leq k \leq 2^n - 1$, set

$$E_{n,k} = \{x \in X; 2^{-n} k \leq f(x) < 2^{-n}(k + 1)\},$$

$$E'_n = \{x \in X; f(x) \geq 2^n\}.$$
These are measurable sets. On increasing \( n \) by one, the interval in the definition of \( E_{n,k} \) is divided into two. It follows that the sequence of simple functions

\[
(3.6) \quad f_n = \sum_k 2^{-n} k \chi_{E_{k,n}} + 2^n \chi_{E_n'}
\]

is increasing and has limit \( f \) and that this limit is uniform on any measurable set where \( f \) is finite. \( \Box \)