Today, we're starting the second unit in this course, which will be Fourier analysis. As an example of how Fourier analysis can be used to solve problems that a priori don't seem to be related to Fourier analysis, let us consider the **Gauss circle problem**. This problem asks us to estimate how many integer lattice points there are in a disk of radius $R$ in $\mathbb{R}^2$. More formally, let

$$N(R) := \# \{(x, y) : x, y \in \mathbb{Z}, (x, y) \in B^2_R \}.$$ 

Then, a reasonable estimate for $N(R)$ is $\pi R^2$, the area of the circle of radius $R$. The error of this estimate is

$$E(R) := N(R) - \pi R^2$$

and what we are interested in is a bound for $|E(R)|$.

First, let us show that we can find some bound for $|E(R)|$.

**Proposition 1.** $|E(R)| \leq 100R$.

**Proof.** For every $v \in \mathbb{Z}^2$, let $Q_v$ be the unit square in $\mathbb{R}^2$ centered at $v$.

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Now,

$$N(R) = \sum_{v \in \mathbb{Z}^2} \chi_{B_R}(v)$$

$$\pi R^2 = \sum_{v \in \mathbb{Z}^2} \text{Area}(Q_v \cap B_R)$$

$$E(R) = N(R) - \pi R^2 = \sum_{v \in \mathbb{Z}^2} (\chi_{B_R}(v) - \text{Area}(Q_v \cap B_R)).$$
But then,
\[ |E(R)| \leq \#\{v : Q_v \cap \partial B_R \neq \emptyset \} \leq \#\{v : Q_v \subset B_{R+3} \setminus B_{R-3} \}. \]

But we could also have cancellation of overestimates and underestimates so it is reasonable to expect that we could get better than a linear bound. For example, in the following picture, the contribution to \( E(R) \) from the shaded box is positive while the contribution from the unshaded boxes is negative. Perhaps we could exploit this cancellation.

![Diagram](image.png)

To get some idea of what bounds on \(|E(R)|\) we might expect to be possible, let us consider a random model. In this random model, \( x_j \in [0, 1] \) are uniformly distributed and independent, \( j = 1, 2, \ldots, N \) \( (N \sim R) \). This represents the contribution to \( E(R) \) of each lattice point where the contribution is nonzero (the points in a distance \( \sqrt{2}/2 \) neighborhood of the circle of radius \( R \)).

**Proposition 2.** \( \mathbb{E}| \sum_{j=1}^N x_j | \leq CN^{1/2} \).

**Proof.**

\[
LHS = \int_{[-1,1]^N} \left| \sum_{j=1}^N x_j \right| \, dx \leq \left( \int_{[-1,1]^N} \left( \sum_{j=1}^N x_j \right)^2 \, dx \right)^{1/2} \\
= \left( \sum_{j_1,j_2} \int_{[-1,1]^2} x_{j_1} x_{j_2} \, dx \right)^{1/2} \\
= \left( \sum_{j=1}^N \int_{[-1,1]} |x_j|^2 \, dx \right)^{1/2} \lesssim N^{1/2}.
\]

Here, we’re using Cauchy Schwarz in the first line and the orthogonality of the \( x_j \) to get the third line.

The conjecture then is that for all \( \epsilon > 0 \), there exists \( C_\epsilon \) such that \( |E(R)| \leq C_\epsilon \cdot R^{1/2+\epsilon} \). What we will prove using tools from Fourier analysis is the following estimate, which is attributed to Sierpinski:
Theorem 3.

\[ |E(R)| \lesssim R^{2/3}. \]

The best current bound of the form \( |E(R)| \lesssim R^c \) is for \( c = 131/208 \approx 0.63 \), proven by Huxley in the early 2000s.

Let us now discuss the Fourier analysis setup in preparation for proving theorem 3. Let

\[ f = \chi_{B_R^2}. \]

And for any \( g \in L^1(\mathbb{R}^d) \), define the **periodization**

\[ Pg(x) = \sum_{v \in \mathbb{Z}^d} g(x + v). \]

Then, \( N(R) = Pf(0) \). If \( g \) is a \( \mathbb{Z}^d \) periodic function on \( \mathbb{R}^d \), then

\[ \hat{g}(n) = \int_{[0,1]^d} g(x) e^{-2\pi in \cdot x} \, dx. \]

We claim now that \( \pi R^2 = \hat{P}f(0) \). This is a result of the Poisson summation formula:

**Theorem 4** (Poisson summation formula). If \( f \in L^1(\mathbb{R}^d) \), \( n \in \mathbb{Z}^d \), then

\[ \hat{P}f(n) = \hat{f}(n) = \int_{\mathbb{R}^d} f(x) e^{-2\pi in \cdot x} \, dx. \]

**Proof.** We have that

\[
\hat{P}f(n) = \int_{[0,1]^d} Pf(x) e^{-2\pi in \cdot x} \, dx \\
= \int_{[0,1]^d} \sum_{v \in \mathbb{Z}^d} f(x + v) e^{-2\pi in \cdot x} \, dx \\
= \sum_{v \in \mathbb{Z}^d} \int_{[0,1]^d} f(x + v) e^{-2\pi in \cdot x} \, dx \\
= \sum_{v \in \mathbb{Z}^d} \int_{[0,1]^d} f(x + v) e^{-2\pi in \cdot (x + v)} \, dx,
\]

since \( n \in \mathbb{Z}^d \). So combining the sum and the integral, we have that

\[
\hat{P}f(n) = \int_{\mathbb{R}^d} f(x) e^{-2\pi in \cdot x} \, dx = \hat{f}(n).
\]

\[ \square \]
Let us do some wishful thinking now. We could wish that
\[ Pf(x) = \sum_{n \in \mathbb{Z}^2} \hat{P}f(n) e^{2\pi in \cdot x}. \]

(But this does not converge pointwise). Then,
\[ N(R) = Pf(0) = \pi R^2 + \sum_{n \neq 0} \hat{P}f(n) \]
and
\[ |E(R)| \leq \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} |\hat{P}f(n)|. \]

Here, we could with that this is \( \leq C \epsilon R^{3/4 + \epsilon} \) (but unfortunately this sum happens to be infinite).

This leads us to the question of when does a Fourier series converge. We can begin to answer this through the following sequence of three theorems, with the first leading to the second leading to the third.

**Theorem 5.** If \( g \in L^2([0,1]^d) \), then \( S_N g \to g \) in \( L^2([0,1]^d) \). Here,
\[ S_N(g) = \sum_{|n| \leq N} \hat{g}(n) e^{2\pi in \cdot x}. \]

**Theorem 6.** If \( g \) is \( C^k \) on \( \mathbb{R}^d \) and \( \mathbb{Z}^d \) periodic, and \( k > n \), then \( S_N g \to g \) uniformly on \( C^0 \).

**Theorem 7.** If \( \sum_n |\hat{g}(n)| < \infty, g \in C^0 \), then \( S_N g \to g \) uniformly in \( C^0 \).

We also have the following question: how can we estimate \( |\hat{g}(n)| \)?

**Proposition 8.** If \( g \) is \( \mathbb{Z}^d \) periodic, \( \|g\|_{C^k} \leq B \), then
\[ |\hat{g}(n)| \leq C(d,k)B \cdot |n|^{-k}. \]

**Proof.** We’ll integrate by parts \( k \) times. For a fixed \( n \), we’ll integrate in \( x_j \) where \( j \) is chosen so that \( |n_j| \leq \frac{1}{d} |n| \). Doing this, we see that
\[
\left| \int_{[0,1]^d} g(x) e^{-2\pi in \cdot x} \, dx \right| = \left| \int_{[0,1]^d} \partial_j g \cdot \frac{1}{-2\pi in_j} e^{2\pi in \cdot x} \, dx \right|
= \left| \int_{[0,1]^d} \partial_j^k g \cdot \frac{1}{(-2\pi in_j)^k} e^{2\pi in \cdot x} \, dx \right|
\leq |n_j|^k \int_{[0,1]^d} |\partial_j^k g| \, dx
\leq |n|^{-k} \|\partial_j^k g\|_{C^0}.
\]
\[ \square \]
As a related question, we might ask if we could have a bound like $|\hat{g}(n)| \lesssim B|n|^{-\alpha}$ if $g \in C^\alpha$. Unfortunately, integration by parts doesn’t work as well here, but we could use another method. Let us define $g_h(x) := g(x - h)$. Then, $|g(x) - g_h(x)| \lesssim h^\alpha$. So,

$$|\hat{g}(n) - \hat{g}_h(n)| = \int (e^{-2\pi inx} - e^{-2\pi in(x+h)})g(x) \, dx = (1 - e^{-2\pi inh})\hat{g}(n).$$

But we also have the bound that

$$|\hat{g}(n) - \hat{g}_h(n)| \leq \int_{[0,1]^d} |g(x) - g(x + h)| \, dx \lesssim h^\alpha.$$ 

Combining these, we have that

$$|\hat{g}(n)| \leq |1 - e^{-2\pi inh}|^{-1}h^\alpha,$$

and we can optimize our choice of $h$ to get the bounds that we want.

Perhaps we’re not satisfied by the integration by parts proof of the previous proposition and want a way of visualizing why smoothness of the function $g$ would lead to decay of the Fourier coefficients $\hat{g}(n)$. Let us consider a smooth, slowly varying function $g$ in one dimension and a large $n$. Then, just looking at the real part for visualization purposes, $\text{Re}(g(x)e^{-2\pi inx})$ looks like a scaled cosine function with some error. The “positive” and “negative” bumps then almost cancel and we would expect more cancellation for larger $n$.

More formally, let us subdivide $[0,1]$ into intervals $I_j$ of length $1/n$. Then,

$$\left| \int_0^1 g(x)e^{-2\pi inx} \, dx \right| = \left| \sum_j \int_{I_j} (g(x) - g(x_j))e^{-2\pi inx} \, dx \right| = \sum_j \int_{I_j} |g(x) - g(x_j)| \, dx,$$

and if $n$ is larger, then we can bound $|g(x) - g(x_j)|$ better.

Our next goal will be to estimate $|\hat{P}f(n)|$. Let us do the first step now. For $f = \chi_{B_R}$,

$$|\hat{P}f(n)| = \left| \int_{B_R} e^{-2\pi inx} \, dx \right|$$

and by rotational invariance, we then have that

$$|\hat{P}f(n)| = \left| \int_{B_R} e^{-2\pi i|n|x_1} \, dx_1 \, dx_2 \right| = \left| \int_{-R}^R 2\sqrt{R^2 - x_1^2}e^{-2\pi i|n|x_1} \, dx_1 \right|.$$