Today’s class will be split up into a discussion of the last problem set and then a continuation of our discussion of Calderon-Zygmund.

1 Pset 4, Problem 3

We’re going to start today with a discussion of problem 3 on the previous homework assignment (problem set 4). A lot of people tried to prove that

\[ V_{T_f}(2^\ell) \lesssim |S_k(f)|^{q_0/p_0 2^{kq_0}} 2^{-\ell q_0} 2^{-\epsilon|k-\ell|}. \]

Unfortunately, this isn’t quite true. Instead, let

\[ A := |S_k(f)|^{q_0/p_0 2^{kq_0}} 2^{-\ell q_0}. \]

We get two bounds from our two \( \|\bar{T}_f\|_{q_i} \lesssim \|f\|_{p_i} \) bounds, and we should let \( \ell \) be the value of \( \ell \) where the two things that we get from these bounds are equal to each other, and use \( \ell \) instead of \( k \). Then,

\[ V_{T_f}(2^\ell) \lesssim A^{2-\epsilon|\ell-\bar{\ell}|} \]

and when \( |\ell - \bar{\ell}| \) is big, we have a gain. We note here that \( \ell \) depends on both \( K \) and \( |S_k(f)| \). We want when

\[ |S_k(f)|^{q_1/p_1 2^{kq_1}} 2^{-\ell q_1} = |S_k(f)|^{q_0/p_0 2^{kq_0}} 2^{-\ell q_0}. \]

We can solve this for \( \bar{\ell} \) if \( q_0 \neq q_1 \). If \( q_1/q_0 \neq p_1/p_0 \), then \( |S_k(f)| \) matters. For the special case when \( q_1 = \infty \), then \( V_{T_{f_k}}(2^\ell) = 0 \) if \( 2^\ell \gg \ldots \) and \( \ell \) is the biggest \( \ell \) consistent with the \( L^\infty \) bound. Now,

\[ \|T_{f_k}\|_{p_0}^{q_0} \sim \sum_\ell V_{T_{f_k}}(2^\ell)^{2^{\ell q_0}} \]

\[ \leq \sum_\ell |S_k(f)|^{q_0/p_0 2^{kq_0}} 2^{-\epsilon|\ell-\bar{\ell}|} \sim \|f_k\|_{p_0}^{q_0}. \]

Now, we want to try to combine all of the \( T_{f_k} \). We have two extreme cases. In the first case, we could have that \( k \mapsto \bar{\ell}(k) \) is injective, in which case we can use weights. In the second case, \( f_k \neq 0 \leftrightarrow k = 1, \ldots, N \) and \( \bar{\ell}(k) = 0 \) for all \( k = 1, \ldots, N \). Then,

\[ \|Tf\|_{q_0} \lesssim \sum_k \|T_{f_k}\|_{q_0} \lesssim \sum_k |S_k(f)|^{1/p_0} 2^k. \]
and we want this \( \lesssim (\sum_k |S_k(f)|2^{k\theta})^{1/p_0} \). But having \( \tilde{\ell}(k) = 0 \) for all \( k = 1, \ldots, N \) gives a formula for \( |S_k(f)| \) and \( |S_k(f)|^{1/p_0}2^k \) gives a geometric series. We get then that

\[
2^\ell = (2^k)^\alpha |S_k(f)|^\beta.
\]

2 Calderon-Zygmund

Let’s go back to the Calderon-Zygmund decomposition lemma. Let us state it again here:

**Lemma 1.** For \( f \in C^0_c, \lambda > 0, \) we can decompose \( f = b + s \), the sum of a balanced part and a small part, such that \( \|b\|_1 + \|s\|_1 \lesssim \|f\|_1 \) and \( \|s\|_\infty \leq \lambda, b = \sum b_j \) where \( b_j \) are balanced for \( \lambda \) supported on disjoint \( Q_j \) and

\[
\int_{Q_j} b_j \lesssim \int_{Q_j} f \lesssim \lambda.
\]

**Proof.** We’re going to use a Calderon-Zygmund iterated stopping time algorithm to construct \( Q_j \) and \( b_j \). Start with a cubical grid in \( \mathbb{R}^d \) of side length \( s \) large and \( \int_{Q_j} |f| < \lambda \) in each cube.

[Call this point in the algorithm (A).] Now, consider each \( Q \).

(i) If \( \int_{Q_j} |f| < \lambda \), subdivide \( Q \) into \( 2^d \) equally sized cubes and repeat this step (A) with each of the subdivided cubes.

(ii) If \( \int_{Q_j} |f| \geq \lambda \), add \( Q \) to the list of balanced cubes, call it \( Q_j \), and let

\[
b_j = f \cdot \chi_{Q_j} - \int_{Q_j} f.
\]

Do not go back to (A) with this cube.

The output of the algorithm is \( \{Q_j\} \) and a function \( b_j \) for each \( Q_j \). Then, let

\[
b = \sum b_j, \quad s = f - b.
\]
We can make some observations now. First,

$$\lambda \leq \int_{Q_j} |f| < 2^d \lambda.$$  

We also have some bound for $s$. If $x \not\in \bigcup Q_j$, then

$$|s(x)| = |f(x)| \leq \lambda.$$  

If $x \in Q_j$, then

$$|s(x)| = |f(x) - b_j(x)| = \left| \frac{1}{|Q_j|} \int_{Q_j} f \right| \leq \frac{1}{|Q_j|} \int_{Q_j} |f| \leq 2^d \lambda,$$

and so we have that

$$\int_{\mathbb{R}^d \setminus \bigcup Q_j} |s| = \int_{\mathbb{R}^d \setminus \bigcup Q_j} |f| \leq \|f\|_{L^1}.$$  

From this, we get that

$$\int_{\bigcup Q_j} |s| \leq \int_{\bigcup Q_j} |f| \leq \|f\|_{L^1},$$

so $\|s\|_1 \leq \|f\|_1$. We also have bounds for the $b_j$:

$$\int_{Q_j} |b_j| = \frac{1}{|Q_j|} \left\| \int_{Q_j} f - \frac{1}{|Q_j|} \int_{Q_j} f \right\| \leq 2 \int_{Q_j} |f|$$

and

$$\int_{Q_j} b_j = \frac{1}{|Q_j|} \int_{Q_j} f - \int_{Q_j} f = 0. \quad \square$$

This lemma then helps us conclude part II of the proof of Calderon-Zygmund, since $V_{Tf}(2\lambda) \leq V_{Ts}(\lambda) + V_{Tb}(\lambda)$. By the $L^2$ bound $V_{Tf}(\lambda) \lesssim \|f\|_1 \lambda^{-1}$. We also have that

$$V_{Tb}(\lambda) \leq \left[ \bigcup_j 2Q_j \right] + \lambda^{-1} \int_{\mathbb{R}^d \setminus \bigcup 2Q_j} |Tb|$$

$$\lesssim \left[ \bigcup_j Q_j \right] + \sum_j \lambda^{-1} \int_{\mathbb{R}^d \setminus 2Q_j} |Tb_j|$$

$$\lesssim \|f\|_1 \lambda^{-1} + \lambda^{-1} \sum_j \|b_j\|_1$$

$$\lesssim \lambda^{-1} (\|s\|_1 + \|b\|_1)$$

$$\lesssim \lambda^{-1} \|f\|_1.$$  

**Part III: Interpolation.** Since we have a weak $L^1$ bound and a strong $L^2$ bound, we can use Marcinkiewicz interpolation to get the bound $\|Tf\|_p \lesssim \|f\|_p$ for $1 < p \leq 2$.  

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