Today, we’ll finish up the proof of the Calderon-Zygmund theorem and see some examples.

**Part IV: Duality.** Recall that we have already proven CZ for $1 < p \leq 2$. Now, let $p$ be such that $2 < p < \infty$ and let $p'$ be the dual exponent (so $1 < p < 2$). Then, we have that

$$\|Tf\|_p = \sup_{\|g\|_{L^{p'}} \leq 1} \int Tf \cdot g$$

$$= \sup \int \int f(y)K(x - y) \, dy \, g(x) \, dx$$

$$= \sup \int f(y) \int K(x - y) g(x) \, dx \, dy$$

$$= \sup \int f \cdot (K \ast g)$$

$$\leq \|f\|_p \cdot \sup \|Tg\|_{L^{p'}}$$

$$\lesssim \|f\|_p,$$

by CZ for $p'$. We note here that $K(x)$ is defined as $K(-x)$ and $Tg = g \ast K$. This completes the proof the the Calderon-Zygmund theorem.

Let us look at an application of CZ now. Suppose that $f_k : \mathbb{R}^d \to \mathbb{C}$,

$$\text{supp } f_k \subset A_k = \{\omega : 2^{k-1} \leq |\omega| \leq 2^{k+1}\},$$

and $f = \sum f_k$. We’ll call this condition (*).
The intuition here is that the \( f_k \) should be almost independent. That is, knowing that \( f_k(x) \in [a, b] \) shouldn’t tell you that much about the value of \( f_\ell(x) \) for \( \ell \neq k \). If the \( f_k \) were indeed independent, then
\[
|\sum f_k| \sim \left( \sum |f_k|^2 \right)^{1/2}
\]
with high probability in \( x \). The next theorem says that our intuition is pretty much what happens.

**Theorem 1** (Littlewood-Paley). If (*) holds, then
\[
\|f\|_{L^p} \sim \left( \sum |f_k|^2 \right)^{1/2}
\]
(up to a factor \( C(p, d) \)).

What we’ll prove today is the \( \lesssim \) in the above theorem. Define
\[
T_k g = (\psi_k \hat{g})^\lor.
\]
Here, \( \psi_k \) is a bump function where \( \psi_k = 1 \) on \( A_k \), and is supported on \( \tilde{A}_k := \{ \omega : 2^{k-2} \leq |\omega| \leq 2^{k+2} \} \). We also want \( \psi_k \) as smooth as possible, and we can show that we can construct \( \psi_k \) so that \( |\psi_k| \leq 1, |\partial \psi_k| \leq 2^{-k}, |\partial^2 \psi_k| \leq 2^{-2k}, \) and so on. We note that \( T_k f_k = f_k \).

Now, we define \( \vec{g} = (\ldots, g_{-1}, g_0, g_1, \ldots) \) and \( \vec{T} \vec{g} = \sum_k T_k g_k \). Note here that \( \vec{T} \vec{f} = f \). Then,
\[
|\vec{g}(x)| = \left( \sum_k |g_k(x)|^2 \right)^{1/2}
\]
and
\[
\|\vec{g}\|_p = \left\| \left( \sum_k |g_k|^2 \right)^{1/2} \right\|_p = \text{RHS of L-P}.
\]

**Theorem 2.** \( \|\vec{T} \vec{g}\|_p \lesssim \|\vec{g}\|_p \) for all \( 1 < p < \infty \) implies the previous theorem (by taking \( \vec{g} = \vec{f} \)).

Notice that \( T_k g_k = g_k \ast \psi_k^\lor \) and
\[
\vec{T} \vec{g} = \vec{g} \ast \vec{\psi}^\lor = \int \vec{g}(y) \cdot \vec{\psi}^\lor(x - y) \, dy = \int \sum_k g_k(x) \psi_k^\lor(x - y) \, dy.
\]

In Calderon-Zygmund, our \( f \) and \( K \) were scalar valued. Now, we have that \( \vec{g}, \vec{K} \) are vector valued in \( \ell^2 \). But this turns out not to be an issue since the proof of Calderon-Zygmund applies almost verbatim for vector valued functions.

So to be able to apply Calderon-Zygmund, we need to check three things:
Proof. Let’s prove each of the three statements above.

(i) We have that \(|\psi_k^\vee(x)| \leq \|\psi_k\|_{L^1} \sim 2^{kd}\) and \(|\psi_k^\vee(x)| \sim 2^{kd}\) for \(|x| \lesssim 2^{-k}\). It also decays rapidly when \(|x| \gg 2^{-k}\) by smoothness (which we can prove by integration by parts). So we have that
\[
\sum_k |\psi_k^\vee(x)|^2 \lesssim \sum_k 2^{2dk} + \text{rapidly decreasing terms} \lesssim |x|^{-2d}.
\]

Now, we get the bound we want by taking square roots.

(ii) We have that \(|\partial \psi_k^\vee(x)| = |(2\pi i \omega \psi_k)^\vee(x)|\). Now, \(|\omega \psi_k| \lesssim 2^k\) and \(|\partial (\omega \psi_k)| \lesssim 1,\ldots\) so \(|\partial \psi_k^\vee(x)| \lesssim 2^{k(d+1)}\) on \(|x| \lesssim 2^{-k}\) and is rapidly decaying for \(|x| \gg 2^{-k}\). Therefore, we have that
\[
\sum_k |\partial \psi_k^\vee(x)|^2 \lesssim \sum_k 2^{2k(d+1)} + \text{rapidly decreasing terms} \lesssim 2^{-2(d+1)}.
\]

Again, we can take square roots to get that bound that we want.

(iii) We have that
\[
\|\vec{T}\vec{g}\|_2^2 = \| \sum_k g_k * \psi_k^\vee \|_2^2 = \| \sum_k \psi_k \cdot \hat{g}_k \|_2^2 = \int \| \sum_k \psi_k \cdot \hat{g}_k \|^2.
\]

Note now that for all \(\omega\), there are \(\leq 4\) nonzero terms in the above sum. So we have that
\[
\|\vec{T}\vec{g}\|_2^2 \lesssim \sum_k |\psi_k \hat{g}_k|^2 \leq \sum_k |\hat{g}_k|^2 = \int \sum_k |g_k|^2 = \|\vec{g}\|_2^2.
\]

Taking square roots give us the bound that we want.

From this lemma and the Calderon-Zygmund theorem, we have the Littlewood-Paley theorem.
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