The Continuity Method

Let $T : B_1 \to B_2$ be linear between two Banach spaces. $T$ is bounded if

$$
||T|| = \sup_{x \in B_1} \frac{||Tx||_{B_2}}{||x||_{B_1}} < \infty \iff ||Tx||_{B_2} \leq c \cdot ||x||_{B_1} \text{ for some } c > 0.
$$

Continuity Method Theorem. Let $B$ be a Banach space, $V$ a normed space, $L_0, L_1 : B \to V$ bounded linear operators. Assume $\exists c$ such that $L_t := (1 - t)L_0 + tL_1$ satisfies

$$
||x||_B \leq c \cdot ||L_t x||_V, \quad \forall t \in [0, 1]. \tag{\ast}
$$

Then – $L_0$ is onto $\iff$ $L_1$ is.

Proof. Assume $L_s$ is onto for some $s \in [0, 1]$; by ($\ast$) $L_s$ is also 1-to-1 $\Rightarrow L_s^{-1}$ exists. For $t \in [0, 1], y \in V$ solving $L_t x = y$ is equivalent to solving $L_s(x) = y + (L_s - L_t)x = y + (t - s)L_0 x + (t - s)L_1 x$.

By linearity now $x = L_s^{-1}y + (t - s)L_s^{-1} \circ (L_0 - L_1)x$.

Define a linear map $T : B \to B$, $Tx = L_s^{-1}y + (t - s)L_s^{-1} \circ (L_0 - L_1)x$. One has $||Tx_1 - Tx_2||_B = ||(t - s)L_s^{-1} \circ (L_0 - L_1)(x_1 - x_2)||$. ($\ast$) now gives us a bound on $L_s^{-1}$: since $L_s$ is onto $\forall x \in B, \exists y \in B$ such that $L_s y = x$ and so

$$
||L_s^{-1}x||_B \leq c \cdot ||L_s \circ L_s^{-1}x||_V
$$

$$
||L_s^{-1}x||_B \leq c \cdot ||x||_V \quad \Rightarrow \quad ||L_s^{-1}|| \leq c.
$$

As an application we see that

$$
||Tx_1 - Tx_2||_B \leq (t - s)c \cdot (||L_0|| + ||L_1||)||x_1 - x_2||,
$$
and for $t$ close enough to $s$ (precisely for $t \in [s - \frac{1}{c(||L_0|| + ||L_1||)}, s + \frac{1}{c(||L_0|| + ||L_1||)}]$) we therefore have a contraction mapping! Therefore $T$ has a fixed point by the previous theorem which essentially means that we can solve $L_t x = y$ for any fixed $y$ or that $L_t$ is onto. Repeating this $c(||L_0|| + ||L_1||)$ many times we cover all $t \in [0, 1]$. ■

Remark. Note as in the beginning of the proof that once such operators are onto they are in fact invertible as long as $(\ast)$ holds.

Elliptic uniqueness

Let us summarize the properties we have established for uniformly elliptic equations. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let $L = a^{ij}(x)D_{ij} + b^i(x)D_i + c(x)$ be uniformly elliptic, i.e

$$\frac{1}{\Lambda} \cdot \delta^{ij} \leq a^{ij}(x) \leq \Lambda \cdot \delta^{ij}$$

and assume $c(x) \leq 0$.

Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of $Lu = f \in C^\alpha(\bar{\Omega})$ with $0 < \alpha < 1$. Then we have the following a priori estimates –

A. $\sup_{\Omega} |u| \leq c(\gamma, \Lambda, \Omega, n) \cdot (\sup_{\partial \Omega} |u| + \sup_{\Omega} |f|)$.

B. Under the additional assumptions

- in the case $L$ has $\alpha$ – Hölder continuous coefficients with Hölder constant $\Lambda$,
- $\Omega$ has $C^{2,\alpha}$ boundary
- $u \in C^{2,\alpha}(\bar{\Omega}), f \in C^\alpha(\bar{\Omega})$,

we had the global Schauder estimate

$$||u||_{C^{2,\alpha}(\bar{\Omega})} \leq c(\gamma, \Lambda, \Omega, n)(||u||_{C^\alpha(\bar{\Omega})} + ||f||_{C^\alpha(\bar{\Omega})}).$$

C. Under the assumptions of B, when $c(x) \leq 0$
\[ \|u\|_{C^{2,\alpha}(\Omega)} \leq c(\sup_{\partial \Omega} |u| + \sup_{\Omega} |f|). \]

D. The above applies to the Dirichlet problem
\[ Lu = f \text{ on } \Omega, \quad u = \varphi \text{ on } \partial \Omega \]
and in particular when \( \varphi = 0 \) we get very simply
\[ \|u\|_{C^{2,\alpha}(\Omega)} \leq c \cdot ||Lu||_{C^\alpha(\Omega)}. \]

**Theorem.** Let \( \Omega \) be a \( C^{2,\alpha} \) domain, \( L \) uniformly elliptic with \( C^\alpha(\overline{\Omega}) \) coefficients and \( (x) \leq 0 \). Look at all \( u \in C^{2,\alpha}(\overline{\Omega}) \) and assume \( f \in C^\alpha(\overline{\Omega}) \). Then the Dirichlet problem \( Lu = f \) on \( \Omega, \ u = \varphi \) on \( \partial \Omega \) has a unique solution \( u \in C^{2,\alpha}(\overline{\Omega}) \) provided that the Dirichlet problem for \( \Delta \) is solvable \( \forall f \in C^\alpha(\overline{\Omega}), \forall \varphi \in C^{2,\alpha}(\overline{\Omega}) \).

**Proof.** Connect \( L \) and \( \Delta \) via a segment: \([0,1] \to L_t := (1-t)L + t\Delta\). Since those operators are all linear it is enough to prove for \( \varphi = 0 \) as we have seen previously. \( C^{2,\alpha}(\overline{\Omega}) \) is a Banach space (Lecture 14), and so is its subspace \( \mathcal{B}(\Omega) := \{u \in C^{2,\alpha}(\overline{\Omega}), u = 0 \text{ on } \partial \Omega\} \). As a matter of fact \( L_t \) is a bounded operator \( \mathcal{B}(\Omega) \to C^\alpha(\overline{\Omega}) \) by the assumptions on the coefficients of \( L \). And, by uniformly elliptic we see from D above
\[ \|u\|_{C^{2,\alpha}(\overline{\Omega})} = \|u\|_{C^{2,\alpha}(\mathcal{B}(\Omega))} \leq c \cdot \|L_t u\|_{C^\alpha(\overline{\Omega})}, \]
with \( c \) independent of \( t \) (depends just on \( L \)). Note \( C^\alpha(\overline{\Omega}) \) is a Banach space and in particular a vector space. The Continuity Method thus applies. \( \blacksquare \)

Strangely enough, we are now back to solving Dirichlet’s problem for \( \Delta \) in domains.

Our methods so far were good for providing solution in balls, spherically symmetric domains. In other words we were able to solve (in \( C^{2,\alpha}(\overline{B(0,R)})! \)) \( \Delta u = f \in C^\alpha(\overline{\Omega}) \) on \( B(0,R), \ u = \varphi \) on \( \partial B(0,R) \) using the Poisson Integral Formula and estimates for the Newtonian Potential. We used conformal mappings (inversion) to get indeed \( C^{2,\alpha} \) upto the boundary. We conclude therefore that
**Corollary.** We can solve the Dirichlet Problem for any $L$ satisfying the assumptions of the Theorem in balls.

Perron’s Method gives a solution in quite general domains but we will not go into its details as later on our regularity theory (weak solutions, Sobolev spaces etc.) will give us those answers.

**Elliptic $C^{2,\alpha}$ regularity**

Let $B := \text{ball}$, $T := \text{some connected boundary portion}$.

**Theorem.** Let $L$ be uniformly elliptic with $C^\alpha$ coefficients and assume $c(x) \leq 0$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of the Dirichlet problem $Lu = f \in C^0(\partial B) \cap C^{2,\alpha}(T)$ on $\partial B$ has a unique solution $u \in C^{2,\alpha}(B \cup T) \cap C^0(\overline{B})$.

We know by the previous theorem that if $\varphi \in C^{2,\alpha}(\partial B)$ (and not just on $T$) then unique solvability would be equivalent to the unique solvability of $\Delta$ on $B$ which we have! Therefore this Theorem is a slight generalization.

**Proof.** As was just outlined the crucial problem lies in the (possible) absence of regularity of $\varphi$ on part of the boundary. So we approximate $\varphi$ by a sequence $\{\varphi_k\} \subset C^3(\overline{B})$ such that both $||\varphi_k - \varphi||_{C^0(\overline{B})} \rightarrow 0$ and $||\varphi_k - \varphi||_{C^{2,\alpha}(\overline{B})} \rightarrow 0$. Solve $Lu_k = f$, in $B$, $u_k = \varphi_k$ on $\partial B$.

Now $L(u_i - u_j) = 0$, in $B$, $u_i - u_j = \varphi_i - \varphi_j$ on $\partial B$. And by A above (as $c(x) \leq 0$) $||u_i - u_j||_{C^0(\overline{B})} \leq C \sup_{\partial B} |\varphi_i - \varphi_j|$. So we conclude our solutions $\{u_k\}$ form a Cauchy sequence wrt the $C^0$ norm, i.e in the Banach space $C^0(B)$. Therefore we know $\exists u \in C^0(B)$ with $u \in C^0(\overline{B})$ (not just subconvergence!) and furthermore this $u$ satisfies $u = \varphi$ on $\partial B$.

Now we shift our look to the $C^{2,\alpha}$ situation; by our interior estimates we have for any $B' \subset B$ $||u_i - u_j||_{C^{2,\alpha}(B')} \leq c(||u_i - u_j||_{C^0(B)} + ||0||_{C^{\alpha}(B)})$.. That is our sequence is also a Cauchy sequence in the Banach space $C^{2,\alpha}(B') \Rightarrow$ converges in $C^{2,\alpha}(B')$ (in particular limit is $C^{2,\alpha}$ regular). This limit must equal the limit $u'_B$ we obtained through the $C^0$ norm. We do this for any $B' \subset B \Rightarrow$ get convergence in $C^{2,\alpha}(B) \Rightarrow u$ satisfies $Lu = f$ on $\overline{B}$ and has the desired $C^{2,\alpha}$ regularity on $B$. 4
We now turn to the boundary portion: \( \forall x_0 \in T \) and \( \rho > 0 \) such that \( B(x_0, \rho) \cap \partial B \subseteq T \) we have the usual boundary Schauder estimates (for smooth enough functions) which give us

\[
\| u_i - u_j \|_{C^{2, \alpha}(B(x_0, \rho) \cap \partial B)} \leq c \cdot (\| u_i - u_j \|_{C^0(B)} + \| \varphi_i - \varphi_j \|_{C^{2, \alpha}(B(x_0, \rho) \cap \partial B)}).
\]

This means that in fact

\[
u_i \overset{C^{2, \alpha}(B(x_0, \rho) \cap \partial B)}{\rightarrow} u \quad \text{and in particular} \quad u \in C^{2, \alpha} \quad \text{at} \quad x_0. \forall x_0 \in T. \quad \blacksquare
\]