18.175: Lecture 10
Zero-one laws and maximal inequalities

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Outline

Recollections

Kolmogorov zero-one law and three-series theorem
Recollections

Kolmogorov zero-one law and three-series theorem
Recall Borel-Cantelli lemmas

- **First Borel-Cantelli lemma**: If $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P(A_n \text{ i.o.}) = 0$.

- **Second Borel-Cantelli lemma**: If $A_n$ are independent, then $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n \text{ i.o.}) = 1$. 
Recall strong law of large numbers

- **Theorem (strong law):** If $X_1, X_2, \ldots$ are i.i.d. real-valued random variables with expectation $m$ and $A_n := n^{-1} \sum_{i=1}^{n} X_i$ are the *empirical means* then $\lim_{n \to \infty} A_n = m$ almost surely.
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Kolmogorov zero-one law and three-series theorem
Consider sequence of random variables $X_n$ on some probability space. Write $\mathcal{F}'_n = \sigma(X_n, X_{n1}, \ldots)$ and $\mathcal{T} = \cap_n \mathcal{F}'_n$.

$\mathcal{T}$ is called the tail $\sigma$-algebra. It contains the information you can observe by looking only at stuff arbitrarily far into the future. Intuitively, membership in tail event doesn’t change when finitely many $X_n$ are changed.

Event that $X_n$ converge to a limit is example of a tail event. Other examples?

**Theorem:** If $X_1, X_2, \ldots$ are independent and $A \in \mathcal{T}$ then $P(A) \in \{0, 1\}$. 
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Main idea of proof: Statement is equivalent to saying that $A$ is independent of itself, i.e., $P(A) = P(A \cap A) = P(A)^2$. How do we prove that?

Recall theorem that if $A_i$ are independent $\pi$-systems, then $\sigma A_i$ are independent.

Deduce that $\sigma(X_1, X_2, \ldots, X_n)$ and $\sigma(X_{n+1}, X_{n+1}, \ldots)$ are independent. Then deduce that $\sigma(X_1, X_2, \ldots)$ and $\mathcal{T}$ are independent, using fact that $\bigcup_k \sigma(X_1, \ldots, X_k)$ and $\mathcal{T}$ are $\pi$-systems.