18.175: Lecture 15

Characteristic functions and central limit theorem

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Outline

Characteristic functions
Characteristic functions
Let $X$ be a random variable.

The characteristic function of $X$ is defined by $\phi(t) = \phi_X(t) := E[e^{itX}]$.

Recall that by definition $e^{it} = \cos(t) + i\sin(t)$.

Characteristic function $\phi_X$ similar to moment generating function $M_X$.

$\phi_{X+Y} = \phi_X \phi_Y$, just as $M_{X+Y} = M_X M_Y$, if $X$ and $Y$ are independent.

And $\phi_{aX}(t) = \phi_X(at)$ just as $M_{aX}(t) = M_X(at)$.

And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.

Characteristic functions are well defined at all $t$ for all random variables $X$. 
Characteristic function properties

- $\phi(0) = 1$
- $\phi(-t) = \overline{\phi(t)}$
- $|\phi(t)| = |E e^{itX}| \leq E|e^{itX}| = 1.$
- $|\phi(t + h) - \phi(t)| \leq E|e^{ihX} - 1|$, so $\phi(t)$ uniformly continuous on $(-\infty, \infty)$
- $E e^{it(aX + b)} = e^{itb} \phi(at)$
Characteristic function examples

- **Coin**: If \( P(X = 1) = P(X = -1) = 1/2 \) then \( \phi_X(t) = (e^{it} + e^{-it})/2 = \cos t \).

- That’s periodic. Do we always have periodicity if \( X \) is a random integer?

- **Poisson**: If \( X \) is Poisson with parameter \( \lambda \) then
  \[
  \phi_X(t) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k e^{itk}}{k!} = \exp(\lambda(e^{it} - 1)).
  \]
  Why does doubling \( \lambda \) amount to squaring \( \phi_X \)?

- **Normal**: If \( X \) is standard normal, then \( \phi_X(t) = e^{-t^2/2} \).

- Is \( \phi_X \) always real when the law of \( X \) is symmetric about zero?

- **Exponential**: If \( X \) is standard exponential (density \( e^{-x} \) on \( (0, \infty) \)) then \( \phi_X(t) = 1/(1 - it) \).

- **Bilateral exponential**: if \( f_X(t) = e^{-|x|}/2 \) on \( \mathbb{R} \) then \( \phi_X(t) = 1/(1 + t^2) \). Use linearity of \( f_X \to \phi_X \).
Fourier inversion formula

- If \( f : \mathbb{R} \to \mathbb{C} \) is in \( L^1 \), write \( \hat{f}(t) := \int_{-\infty}^{\infty} f(x)e^{-itx} dx \).
- **Fourier inversion**: If \( f \) is nice: \( f(x) = \frac{1}{2\pi} \int \hat{f}(t)e^{itx} dt \).
- Easy to check this when \( f \) is density function of a Gaussian. Use linearity of \( f \to \hat{f} \) to extend to linear combinations of Gaussians, or to convolutions with Gaussians.
- Show \( f \to \hat{f} \) is an isometry of Schwartz space (endowed with \( L^2 \) norm). Extend definition to \( L^2 \) completion.
- **Convolution theorem**: If

\[
h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy,
\]

then

\[
\hat{h}(t) = \hat{f}(t)\hat{g}(t).
\]

- **Possible application?**

\[
\int 1_{[a,b]}(x)f(x)dx = (\widehat{1_{[a,b]}f})(0) = (\hat{f} * \widehat{1_{[a,b]}})(0) = \int \hat{f}(t)\widehat{1_{[a,b]}(-t)}dx.
\]
Characteristic function inversion formula

- If the map $\mu_X \to \phi_X$ is linear, is the map $\phi \to \mu[a, b]$ (for some fixed $[a, b]$) a linear map? How do we recover $\mu[a, b]$ from $\phi$?
- Say $\phi(t) = \int e^{itx} \mu(x)$.
- **Inversion theorem:**

$$\lim_{T \to \infty} (2\pi)^{-1} \int_{-T}^{T} \frac{e^{-ita} - e^{itb}}{it} \phi(t) dt = \mu(a, b) + \frac{1}{2} \mu([a, b])$$

- **Main ideas of proof:** Write

$$I_T = \int \frac{e^{-ita} - e^{-itb}}{it} \phi(t) dt = \int_{-T}^{T} \int \frac{e^{-ita} - e^{-itb}}{it} e^{itx} \mu(x) dt.$$ 

- Observe that $\frac{e^{-ita} - e^{-itb}}{it} = \int_a^b e^{-ity} dy$ has modulus bounded by $b - a$.
- That means we can use Fubini to compute $I_T$. 

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Bochner’s theorem

- Given any function $\phi$ and any points $x_1, \ldots, x_n$, we can consider the matrix with $i, j$ entry given by $\phi(x_i - x_j)$. Call $\phi$ **positive definite** if this matrix is always positive semidefinite Hermitian.

- Bochner’s theorem: a continuous function from $\mathbb{R}$ to $\mathbb{R}$ with $\phi(1) = 1$ is a characteristic function of a some probability measure on $\mathbb{R}$ if and only if it is positive definite.

- Positive definiteness kind of comes from fact that variances of random variables are non-negative.

- The set of all possible characteristic functions is a pretty nice set.
Continuity theorems

- **Lévy’s continuity theorem**: if

\[
\lim_{n \to \infty} \phi_{X_n}(t) = \phi_X(t)
\]

for all \( t \), then \( X_n \) converge in law to \( X \).

- **Slightly stronger theorem**: If \( \mu_n \Rightarrow \mu_\infty \) then \( \phi_n(t) \to \phi_\infty(t) \) for all \( t \). Conversely, if \( \phi_n(t) \) converges to a limit that is continuous at 0, then the associated sequence of distributions \( \mu_n \) is tight and converges weakly to measure \( \mu \) with characteristic function \( \phi \).

- **Proof ideas**: First statement easy (since \( X_n \Rightarrow X \) implies \( Eg(X_n) \to Eg(X) \) for any bounded continuous \( g \)). To get second statement, first play around with Fubini and establish tightness of the \( \mu_n \). Then note that any subsequential limit of the \( \mu_n \) must be equal to \( \mu \). Use this to argue that \( \int fd\mu_n \) converges to \( \int fd\mu \) for every bounded continuous \( f \).
Moments, derivatives, CLT

- If $\int |x|^n \mu(x) < \infty$ then the characteristic function $\phi$ of $\mu$ has a continuous derivative of order $n$ given by
  \[ \phi^{(n)}(t) = \int (ix)^n e^{itx} \mu(dx). \]

- Indeed, if $E|X|^2 < \infty$ and $EX = 0$ then
  \[ \phi(t) = 1 - t^2 E(X^2)/2\sigma(t^2). \]

- This and the continuity theorem together imply the central limit theorem.

- **Theorem:** Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = \mu$, $\text{Var}(X_i) = \sigma^2 \in (0, \infty)$. If $S_n = X_1 + \ldots + X_n$ then
  \[ (S_n - n\mu)/(\sigma n^{1/2}) \]
  converges in law to a standard normal.