18.175: Lecture 18
Poisson random variables

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Extend CLT idea to stable random variables
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Recall continuity theorem

- **Strong continuity theorem:** If $\mu_n \Rightarrow \mu_\infty$ then $\phi_n(t) \to \phi_\infty(t)$ for all $t$. Conversely, if $\phi_n(t)$ converges to a limit that is continuous at 0, then the associated sequence of distributions $\mu_n$ is tight and converges weakly to a measure $\mu$ with characteristic function $\phi$. 
Recall CLT idea

- Let $X$ be a random variable.
- The **characteristic function** of $X$ is defined by
  \[ \phi(t) = \phi_X(t) := E[e^{itX}] . \]
- And if $X$ has an $m$th moment then $E[X^m] = i^m \phi_X^{(m)}(0)$.
- In particular, if $E[X] = 0$ and $E[X^2] = 1$ then $\phi_X(0) = 1$ and $\phi_X'(0) = 0$ and $\phi_X''(0) = -1$.
- Write $L_X := -\log \phi_X$. Then $L_X(0) = 0$ and
  \[ L_X'(0) = -\phi_X'(0)/\phi_X(0) = 0 \quad \text{and} \quad L_X'' = -\frac{(\phi_X''(0)\phi_X(0) - \phi_X'(0)^2)}{\phi_X(0)^2} = 1. \]
- If $V_n = n^{-1/2} \sum_{i=1}^n X_i$ where $X_i$ are i.i.d. with law of $X$, then $L_{V_n}(t) = nL_X(n^{-1/2}t)$.
- When we zoom in on a twice differentiable function near zero (scaling vertically by $n$ and horizontally by $\sqrt{n}$) the picture looks increasingly like a parabola.
Question? Is it possible for something like a CLT to hold if $X$ has infinite variance? Say we write $V_n = n^{-a} \sum_{i=1}^{n} X_i$ for some $a$. Could the law of these guys converge to something non-Gaussian?

What if the $L_{V_n}$ converge to something else as we increase $n$, maybe to some other power of $|t|$ instead of $|t|^2$?

The the appropriately normalized sum should be converge in law to something with characteristic function $e^{-|t|\alpha}$ instead of $e^{-|t|^2}$.

We already saw that this should work for Cauchy random variables.
Example: Suppose that $P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2$ for $0 < \alpha < 2$. This is a random variable with a “power law tail”.

Compute $1 - \phi(t) \approx C|t|^\alpha$ when $|t|$ is large.

If $X_1, X_2, \ldots$ have same law as $X_1$ then we have $E \exp(itS_n/n^{1/\alpha}) = \phi(t/n^\alpha)^n = (1 - (1 - \phi(t/n^{1/\alpha})))$. As $n \to \infty$, this converges pointwise to $\exp(-C|t|^{\alpha})$.

Conclude by continuity theorems that $X_n/n^{1/\alpha} \implies Y$ where $Y$ is a random variable with $\phi_Y(t) = \exp(-C|t|^\alpha)$

Let’s look up stable distributions. Up to affine transformations, this is just a two-parameter family with characteristic functions $\exp[-|t|^\alpha(1 - i\beta \text{sgn}(t)\Phi)]$ where $\Phi = \tan(\pi\alpha/2)$ where $\beta \in [-1, 1]$ and $\alpha \in (0, 2]$. 
Stable-Poisson connection

Let’s think some more about this example, where
\[ P(X_1 > x) = P(X_1 < -x) = x^{-\alpha}/2 \] for \( 0 < \alpha < 2 \) and
\( X_1, X_2, \ldots \) are i.i.d.

Now \( P(an^{1/\alpha} < X_1 < bn^{1/\alpha}) = \frac{1}{2}(a^{-\alpha} - b^{-\alpha})n^{-1} \).

So \( \{m \leq n : X_m/n^{1/\alpha} \in (a, b)\} \) converges to a Poisson distribution with mean \( (a^{-\alpha} - b^{-\alpha})/2 \).

More generally \( \{m \leq n : X_m/n^{1/\alpha} \in (a, b)\} \) converges in law to Poisson with mean \( \int_A \frac{x^\alpha}{2|x|^{\alpha+1}} \, dx < \infty \).
More generality: suppose that
\[ \lim_{x \to \infty} \frac{P(X_1 > x)}{P(|X_1| > x)} = \theta \in [0, 1] \] and
\[ P(|X_1| > x) = x^{-\alpha}L(x) \] where \( L \) is slowly varying (which means \( \lim_{x \to \infty} L(tx)/L(x) = 1 \) for all \( t > 0 \)).

**Theorem:** Then \((S_n - b_n)/a_n\) converges in law to limiting random variable, for appropriate \(a_n\) and \(b_n\) values.
Infinitely divisible laws

Say a random variable $X$ is **infinitely divisible**, for each $n$, there is a random variable $Y$ such that $X$ has the same law as the sum of $n$ i.i.d. copies of $Y$.

What random variables are infinitely divisible?
- Poisson, Cauchy, normal, stable, etc.

Let’s look at the characteristic functions of these objects. What about compound Poisson random variables (linear combinations of Poisson random variables)? What are their characteristic functions like?

More general constructions are possible via Lévy Khintchine representation.
Higher dimensional limit theorems

- Much of the CLT story generalizes to higher dimensional random variables.
- For example, given a random vector \((X, Y, Z)\), we can define 
  \[ \phi(a, b, c) = E e^{i(aX + bY + cZ)}. \]
- This is just a higher dimensional Fourier transform of the density function.
- The inversion theorems and continuity theorems that apply here are essentially the same as in the one-dimensional case.