18.175: Lecture 30
Markov chains

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Outline

Review what you know about finite state Markov chains

Finite state ergodicity and stationarity

More general setup
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More general setup
Consider a sequence of random variables $X_0, X_1, X_2, \ldots$ each taking values in the same state space, which for now we take to be a finite set that we label by $\{0, 1, \ldots, M\}$.

Interpret $X_n$ as state of the system at time $n$.

Sequence is called a **Markov chain** if we have a fixed collection of numbers $P_{ij}$ (one for each pair $i, j \in \{0, 1, \ldots, M\}$) such that whenever the system is in state $i$, there is probability $P_{ij}$ that system will next be in state $j$.

Precisely,
\[
P\{X_{n+1} = j|X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P_{ij}.
\]

Kind of an “almost memoryless” property. Probability distribution for next state depends only on the current state (and not on the rest of the state history).
Simple example

- For example, imagine a simple weather model with two states: rainy and sunny.
- If it’s rainy one day, there’s a .5 chance it will be rainy the next day, a .5 chance it will be sunny.
- If it’s sunny one day, there’s a .8 chance it will be sunny the next day, a .2 chance it will be rainy.
- In this climate, sun tends to last longer than rain.
- Given that it is rainy today, how many days to I expect to have to wait to see a sunny day?
- Given that it is sunny today, how many days to I expect to have to wait to see a rainy day?
- Over the long haul, what fraction of days are sunny?
To describe a Markov chain, we need to define $P_{ij}$ for any $i, j \in \{0, 1, \ldots, M\}$.

It is convenient to represent the collection of transition probabilities $P_{ij}$ as a matrix:

$$A = \begin{pmatrix}
P_{00} & P_{01} & \cdots & P_{0M} \\
P_{10} & P_{11} & \cdots & P_{1M} \\
\vdots & \vdots & \ddots & \vdots \\
P_{M0} & P_{M1} & \cdots & P_{MM}
\end{pmatrix}$$

For this to make sense, we require $P_{ij} \geq 0$ for all $i, j$ and $\sum_{j=0}^{M} P_{ij} = 1$ for each $i$. That is, the rows sum to one.
Transitions via matrices

- Suppose that \( p_i \) is the probability that system is in state \( i \) at time zero.
- What does the following product represent?

\[
\begin{pmatrix}
p_0 & p_1 & \ldots & p_M \\
\end{pmatrix}
\begin{pmatrix}
P_{00} & P_{01} & \ldots & P_{0M} \\
P_{10} & P_{11} & \ldots & P_{1M} \\
\vdots & \vdots & \ddots & \vdots \\
P_{M0} & P_{M1} & \ldots & P_{MM} \\
\end{pmatrix}
\]

- Answer: the probability distribution at time one.
- How about the following product?

\[
\begin{pmatrix}
p_0 & p_1 & \ldots & p_M \\
\end{pmatrix}A^n
\]

- Answer: the probability distribution at time \( n \).
Powers of transition matrix

- We write $P_{ij}^{(n)}$ for the probability to go from state $i$ to state $j$ over $n$ steps.

- From the matrix point of view

$$\begin{pmatrix}
P_{00}^{(n)} & P_{01}^{(n)} & \ldots & P_{0M}^{(n)} \\
P_{10}^{(n)} & P_{11}^{(n)} & \ldots & P_{1M}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
P_{M0}^{(n)} & P_{M1}^{(n)} & \ldots & P_{MM}^{(n)}
\end{pmatrix} = \begin{pmatrix}
P_{00} & P_{01} & \ldots & P_{0M} \\
P_{10} & P_{11} & \ldots & P_{1M} \\
\vdots & \vdots & \ddots & \vdots \\
P_{M0} & P_{M1} & \ldots & P_{MM}
\end{pmatrix}^n$$

- If $A$ is the one-step transition matrix, then $A^n$ is the $n$-step transition matrix.
Questions

- What does it mean if all of the rows are identical?
  - Answer: state sequence $X_i$ consists of i.i.d. random variables.
- What if matrix is the identity?
  - Answer: states never change.
- What if each $P_{ij}$ is either one or zero?
  - Answer: state evolution is deterministic.
Consider the simple weather example: If it’s rainy one day, there’s a .5 chance it will be rainy the next day, a .5 chance it will be sunny. If it’s sunny one day, there’s a .8 chance it will be sunny the next day, a .2 chance it will be rainy.

Let rainy be state zero, sunny state one, and write the transition matrix by

$$A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix}$$

Note that

$$A^2 = \begin{pmatrix} .64 & .35 \\ .26 & .74 \end{pmatrix}$$

Can compute $A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix}$
Does relationship status have the Markov property?

- Can we assign a probability to each arrow?
- Markov model implies time spent in any state (e.g., a marriage) before leaving is a geometric random variable.
- Not true... Can we make a better model with more states?
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Ergodic Markov chains

- Say Markov chain is **ergodic** if some power of the transition matrix has all non-zero entries.
- Turns out that if chain has this property, then 
  \[ \pi_j := \lim_{n \to \infty} P_{ij}^{(n)} \]
  exists and the \( \pi_j \) are the unique non-negative solutions of \( \pi_j = \sum_{k=0}^{M} \pi_k P_{kj} \) that sum to one.
- This means that the row vector
  \[
  \pi = \left( \begin{array}{c}
  \pi_0 \\
  \pi_1 \\
  \vdots \\
  \pi_M 
  \end{array} \right)
  \]
  is a left eigenvector of \( A \) with eigenvalue 1, i.e., \( \pi A = \pi \).
- We call \( \pi \) the **stationary distribution** of the Markov chain.

- One can solve the system of linear equations
  \[ \pi_j = \sum_{k=0}^{M} \pi_k P_{kj} \]
  to compute the values \( \pi_j \). Equivalent to considering \( A \) fixed and solving \( \pi A = \pi \). Or solving \( (A - I)\pi = 0 \). This determines \( \pi \) up to a multiplicative constant, and fact that \( \sum \pi_j = 1 \) determines the constant.
Simple example

- If \( A = \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} \), then we know

\[
\pi A = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \end{pmatrix} = \pi.
\]

- This means that \(.5\pi_0 + .2\pi_1 = \pi_0\) and \(.5\pi_0 + .8\pi_1 = \pi_1\) and we also know that \(\pi_1 + \pi_2 = 1\). Solving these equations gives \(\pi_0 = 2/7\) and \(\pi_1 = 5/7\), so \(\pi = \begin{pmatrix} 2/7 \\ 5/7 \end{pmatrix}\).

- Indeed,

\[
\pi A = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix} \begin{pmatrix} .5 & .5 \\ .2 & .8 \end{pmatrix} = \begin{pmatrix} 2/7 & 5/7 \end{pmatrix} = \pi.
\]

- Recall that

\[
A^{10} = \begin{pmatrix} .285719 & .714281 \\ .285713 & .714287 \end{pmatrix} \approx \begin{pmatrix} 2/7 & 5/7 \\ 2/7 & 5/7 \end{pmatrix} = \begin{pmatrix} \pi \\ \pi \end{pmatrix}
\]
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Consider a measurable space \((S, S)\).

A function \(p : S \times S \to \mathbb{R}\) is a transition probability if

- For each \(x \in S\), \(A \to p(x, A)\) is a probability measure on \((S, S)\).
- For each \(A \in S\), the map \(x \to p(x, A)\) is a measurable function.

Say that \(X_n\) is a Markov chain w.r.t. \(F_n\) with transition probability \(p\) if
\[
P(X_{n+1} \in B|F_n) = p(X_n, B).
\]

How do we construct an infinite Markov chain? Choose \(p\) and initial distribution \(\mu\) on \((S, S)\). For each \(n < \infty\) write
\[
P(X_j \in B_j, 0 \leq j \leq n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_{n-1}} p(x_{n-1}, dx_n).
\]

Extend to \(n = \infty\) by Kolmogorov’s extension theorem.
Markov chains

- **Definition, again:** Say $X_n$ is a Markov chain w.r.t. $\mathcal{F}_n$ with transition probability $p$ if $P(X_{n+1} \in B|\mathcal{F}_n) = p(X_n, B)$.

- **Construction, again:** Fix initial distribution $\mu$ on $(S, S)$. For each $n < \infty$ write

  $$P(X_j \in B_j, 0 \leq j \leq n) = \int_{B_0} \mu(dx_0) \int_{B_1} p(x_0, dx_1) \cdots \int_{B_n} p(x_{n-1}, dx_n).$$

  Extend to $n = \infty$ by Kolmogorov’s extension theorem.

- **Notation:** Extension produces probability measure $P_\mu$ on sequence space $(S^{0,1}, \ldots, S^{0,1}, \ldots)$.

- **Theorem:** $(X_0, X_1, \ldots)$ chosen from $P_\mu$ is Markov chain.

- **Theorem:** If $X_n$ is any Markov chain with initial distribution $\mu$ and transition $p$, then finite dim. probabilities are as above.
Examples

- Random walks on $\mathbb{R}^d$.
- Branching processes: $p(i, j) = P(\sum_{m=1}^{i} \xi_m = j)$ where $\xi_i$ are i.i.d. non-negative integer-valued random variables.
- Renewal chain.
- Card shuffling.
- Ehrenfest chain.