18.175: Lecture 4

Integration

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Outline

Integration

Expectation
Integration

Expectation
Recall definitions

- **Probability space** is triple \((\Omega, \mathcal{F}, \mathcal{P})\) where \(\Omega\) is sample space, \(\mathcal{F}\) is set of events (the \(\sigma\)-algebra) and \(\mathcal{P} : \mathcal{F} \rightarrow [0, 1]\) is the probability function.

- **\(\sigma\)-algebra** is collection of subsets closed under complementation and countable unions. Call \((\Omega, \mathcal{F})\) a measure space.

- **Measure** is function \(\mu : \mathcal{F} \rightarrow \mathbb{R}\) satisfying \(\mu(A) \geq \mu(\emptyset) = 0\) for all \(A \in \mathcal{F}\) and countable additivity: \(\mu(\bigcup_i A_i) = \sum_i \mu(A_i)\) for disjoint \(A_i\).

- Measure \(\mu\) is **probability measure** if \(\mu(\Omega) = 1\).

- The **Borel \(\sigma\)-algebra** \(\mathcal{B}\) on a topological space is the smallest \(\sigma\)-algebra containing all open sets.
Recall definitions

- Real random variable is function $X : \Omega \rightarrow \mathbb{R}$ such that the preimage of every Borel set is in $\mathcal{F}$.
- Note: to prove $X$ is measurable, it is enough to show that the pre-image of every open set is in $\mathcal{F}$.
- Can talk about $\sigma$-algebra generated by random variable(s): smallest $\sigma$-algebra that makes a random variable (or a collection of random variables) measurable.
Lebesgue integration

- Lebesgue: If you can measure, you can integrate.
- In more words: if \((\Omega, \mathcal{F})\) is a measure space with a measure \(\mu\) with \(\mu(\Omega) < \infty\) and \(f : \Omega \rightarrow \mathbb{R}\) is \(\mathcal{F}\)-measurable, then we can define \(\int f d\mu\) (for non-negative \(f\), also if both \(f \vee 0\) and \(-f \wedge 0\) and have finite integrals...)
- Idea: define integral, verify linearity and positivity (a.e. non-negative functions have non-negative integrals) in 4 cases:
  - \(f\) takes only finitely many values.
  - \(f\) is bounded (hint: reduce to previous case by rounding down or up to nearest multiple of \(\epsilon\) for \(\epsilon \rightarrow 0\)).
  - \(f\) is non-negative (hint: reduce to previous case by taking \(f \wedge N\) for \(N \rightarrow \infty\)).
  - \(f\) is any measurable function (hint: treat positive/negative parts separately, difference makes sense if both integrals finite).
Can we extend previous discussion to case $\mu(\Omega) = \infty$?

**Theorem:** if $f$ and $g$ are integrable then:

- If $f \geq 0$ a.s. then $\int f d\mu \geq 0$.
- For $a, b \in \mathbb{R}$, have $\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu$.
- If $g \leq f$ a.s. then $\int g d\mu \leq \int f d\mu$.
- If $g = f$ a.e. then $\int g d\mu = \int f d\mu$.
- $|\int f d\mu| \leq \int |f| d\mu$.

- When $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{B}^d, \lambda)$, write $\int_E f(x) dx = \int 1_E f d\lambda$. 
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