18.175: Lecture 5

More integration and expectation

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Outline

Integration

Expectation
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Integration

Expectation
Recall Lebesgue integration

- Lebesgue: If you can measure, you can integrate.

- In more words: if \((\Omega, \mathcal{F})\) is a measure space with a measure \(\mu\) with \(\mu(\Omega) < \infty\) and \(f : \Omega \to \mathbb{R}\) is \(\mathcal{F}\)-measurable, then we can define \(\int f d\mu\) (for non-negative \(f\), also if both \(f \lor 0\) and \(-f \land 0\) and have finite integrals...)

- Idea: define integral, verify linearity and positivity (a.e. non-negative functions have non-negative integrals) in 4 cases:
  - \(f\) takes only finitely many values.
  - \(f\) is bounded (hint: reduce to previous case by rounding down or up to nearest multiple of \(\epsilon\) for \(\epsilon \to 0\)).
  - \(f\) is non-negative (hint: reduce to previous case by taking \(f \land N\) for \(N \to \infty\)).
  - \(f\) is any measurable function (hint: treat positive/negative parts separately, difference makes sense if both integrals finite).
Theorem: if $f$ and $g$ are integrable then:
- If $f \geq 0$ a.s. then $\int fd\mu \geq 0$. 
- For $a, b \in \mathbb{R}$, have $\int (af + bg)d\mu = a\int fd\mu + b\int gd\mu$.
- If $g \leq f$ a.s. then $\int gd\mu \leq \int fd\mu$.
- If $g = f$ a.e. then $\int gd\mu = \int fd\mu$.
- $|\int fd\mu| \leq \int |f|d\mu$.

When $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{B}^d, \lambda)$, write $\int_E f(x)dx = \int 1_E fd\lambda$. 

Lebesgue integration
Outline

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Expectation
Given probability space \((\Omega, \mathcal{F}, P)\) and random variable \(X\), we write \(EX = \int XdP\). Always defined if \(X \geq 0\), or if integrals of \(\max\{X, 0\}\) and \(\min\{X, 0\}\) are separately finite.

- \(EX^k\) is called \(k\)th moment of \(X\). Also, if \(m = EX\) then \(E(X - m)^2\) is called the variance of \(X\).
Properties of expectation/integration

- **Jensen’s inequality:** If $\mu$ is a probability measure and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is convex then $\phi(\int f \, d\mu) \leq \int \phi(f) \, d\mu$. If $X$ is a random variable then $E\phi(X) \geq \phi(EX)$.

- **Main idea of proof:** Approximate $\phi$ below by linear function $L$ that agrees with $\phi$ at $EX$.

- **Applications:** Utility, hedge fund payout functions.

- **Hölder’s inequality:** Write $\|f\|_p = \left( \int |f|^p \, d\mu \right)^{1/p}$ for $1 \leq p < \infty$. If $1/p + 1/q = 1$, then $\int |fg| \, d\mu \leq \|f\|_p \|g\|_q$.

- **Main idea of proof:** Rescale so that $\|f\|_p \|g\|_q = 1$. Use some basic calculus to check that for any positive $x$ and $y$ we have $xy \leq x^p/p + y^q/q$. Write $x = |f|$, $y = |g|$ and integrate to get $\int |fg| \, d\mu \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q$.

- **Cauchy-Schwarz inequality:** Special case $p = q = 2$. Gives $\int |fg| \, d\mu \leq \|f\|_2 \|g\|_2$. Says that dot product of two vectors is at most product of vector lengths.
Bounded convergence theorem: Consider probability measure \( \mu \) and suppose \( |f_n| \leq M \) a.s. for all \( n \) and some fixed \( M > 0 \), and that \( f_n \to f \) in probability (i.e., 
\[
\lim_{n \to \infty} \mu \{ x : |f_n(x) - f(x)| > \epsilon \} = 0 \text{ for all } \epsilon > 0.
\]
Then
\[
\int fd\mu = \lim_{n \to \infty} \int f_n d\mu.
\]

(Build counterexample for infinite measure space using wide and short rectangles?...)

Main idea of proof: for any \( \epsilon, \delta \) can take \( n \) large enough so
\[
\int |f_n - f| d\mu < M\delta + \epsilon.
\]
Fatou’s lemma

- **Fatou’s lemma**: If $f_n \geq 0$ then

$$\liminf_{n \to \infty} f_n d\mu \geq (\liminf_{n \to \infty} f_n) d\mu.$$ 

(Counterexample for opposite-direction inequality using thin and tall rectangles?)

- **Main idea of proof**: first reduce to case that the $f_n$ are increasing by writing $g_n(x) = \inf_{m \geq n} f_m(x)$ and observing that $g_n(x) \uparrow g(x) = \liminf_{n \to \infty} f_n(x)$. Then truncate, used bounded convergence, take limits.
More integral properties

- **Monotone convergence:** If \( f_n \geq 0 \) and \( f_n \uparrow f \) then
  \[
  \int f_n d\mu \uparrow \int f d\mu.
  \]

- **Main idea of proof:** one direction obvious, Fatou gives other.

- **Dominated convergence:** If \( f_n \to f \) a.e. and \( |f_n| \leq g \) for all \( n \) and \( g \) is integrable, then \( \int f_n d\mu \to \int f d\mu \).

- **Main idea of proof:** Fatou for functions \( g + f_n \geq 0 \) gives one side. Fatou for \( g - f_n \geq 0 \) gives other.