Problem 1. Show that the number of non-crossing partitions of the set \( \{1, \ldots, n\} \) equals the Catalan number \( C_n = \frac{1}{n+1} \binom{2n}{n} \). (A bijective proof is preferable. For example, you can use the fact that \( C_n \) is equal to the number of Dyck paths with \( 2n \) steps.)

Problem 2. (a) Prove the recurrence relation for the signless Stirling numbers of the first kind
\[
c(n + 1, k) = n c(n, k) + c(n, k - 1).
\]

(b) Prove the recurrence relation for the Stirling numbers of the second kind:
\[
S(n + 1, k) = k S(n, k) + S(n, k - 1).
\]

Problem 3. The Bell number \( B(n) \) is the total number of partitions of an \( n \) element set, i.e., \( B(n) = S(n, 1) + S(n, 2) + \cdots + S(n, n) \).

Show that the Bell numbers can be calculated using the Bell triangle:

\[
\begin{array}{cccccc}
1 & 1 & 2 & 3 & 5 & 7 \\
2 & 3 & 5 & 9 & 15 & 20 \\
3 & 5 & 10 & 15 & 30 & 37 \\
4 & 7 & 10 & 27 & 52 & 75 \\
5 & 9 & 15 & 27 & 37 & 72 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

In this triangle, the first number in each row (except the first row) equals the last number in the previous row; and any other number equals the sum of the two numbers to the left and above it. The Bell numbers \( B(0) = 1, B(1) = 1, B(2) = 2, B(3) = 5, B(4) = 15, B(5) = 52 \ldots \) appear as the first entries (and also the last entries) in rows of this triangle.

Problem 4. Show that the Bell number \( B(n) \) is given by
\[
B(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.
\]
Problem 5. In class, we mentioned two ways to define a lattice.

(I) A set $L$ with two binary operation called “meet” $\lor$ and “join” $\land$ that satisfy several axioms.

(II) A poset $P$ such that, for any two elements $x, y \in P$, there is a unique minimal element $u$ such that $u \geq x$ and $u \geq y$, and a unique maximal element $v$ such that $v \leq x$ and $v \leq y$.

Show that these two definitions of lattices are equivalent.

Problem 6. Let $L$ be a finite distributive lattice. Let $P$ be the poset formed by all join-irreducible elements of $L$. Use axioms of distributive lattices to show that $L$ is isomorphic to $J(P)$.

Problem 7. Let $P$ be a finite poset. Prove Dilworth’s theorem that claims that the maximal size $M(P)$ of an anti-chain in $P$ equals the minimal number $m(P)$ of disjoint chains (not necessarily saturated) that cover all elements of $P$.

Problem 8. (a) Show that the Fibonacci number $F_{n+1}$ equals the number of compositions of $n$ with all parts equal to 1 or 2, that is, the number of ordered sequences $c_1 \ldots c_l$ such that $c_1 + \cdots + c_l = n$ and all $c_i \in \{1, 2\}$. For example,

$$F_6 = \#\{11111, 1112, 1121, 1211, 2111, 122, 212, 221\} = 8.$$  

(b) In class, we gave a recursive construction of the differential poset $\mathbb{F}$ called the Fibonacci lattice. Give a nonrecursive description of $\mathbb{F}$ as a certain order relation on compositions with parts equal to 1 or 2.

(c) Prove that $\mathbb{F}$ is indeed a lattice.

Problem 9. Let $W_n$ be the number of walks with $2n$ steps on the Hasse diagram of the Young’s lattice $\mathbb{Y}$ that start and end at the minimal element $\hat{0} = (0)$. (The walks can have up and down steps in any order.)

For example, $W_2 = 3$, because there are 3 walks with 4 steps:

$$(0) \to (1) \to (2) \to (1) \to (0)$$
$$(0) \to (1) \to (1, 1) \to (1) \to (0)$$
$$(0) \to (1) \to (0) \to (1) \to (0)$$

Show that $W_n$ equals the number of perfect matchings in the complete graph $K_{2n}$. Find a closed formula for $W_n$. 
Problem 10. Let $X$ and $D$ be two operators that act on polynomials $f(x)$ as follows:

$$X : f(x) \mapsto xf(x) \quad \text{and} \quad D : f(x) \mapsto f'(x).$$

For $n \geq 0$, define the polynomials $f_n(x) := (X + D)^n(1)$. For example, $f_0 = 1$, $f_1 = x$, $f_2 = x^2 + 1$, $f_3 = x^3 + 3x$. Calculate the constant term $f_n(0)$ of the polynomial $f_n$.

Problem 11. Fix positive integers $k$ and $l$. Define the weight function $w(x)$ on boxes $x = (i, j)$ of the $k \times l$ rectangular Young diagram by

$$w((i, j)) := (i - j + l)(j - i + k),$$

for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$.

Show that, for any Young diagram $\lambda$ that fits inside the $k \times l$ rectangle, we have

$$\sum_{x \in \text{Add}(\lambda)} w(x) - \sum_{y \in \text{Remove}(\lambda)} w(y) = k \cdot l - 2 |\lambda|.$$

Here $\text{Add}(\lambda)$ is the set of all boxes of the $k \times l$ rectangle that can be added to the Young diagram $\lambda$; and $\text{Remove}(\lambda)$ is the set of all boxes that can be removed from $\lambda$.

Problem 12. Show that the poset $J(J([2] \times [n]))$ is unimodal. (This is the poset of all shifted Young diagrams that fit inside the shifted shape $(n, n - 1, \ldots, 1)$ ordered by containment.)

Problem 13. Find a closed formula for the number of saturated chains from the minimal element $\hat{0}$ to the maximal element $\hat{1}$ in the partition lattice $\Pi_n$.

Problem 14. Let $NC_n$ be the subposet of the partition lattice $\Pi_n$ formed by all non-crossing partitions of the set $\{1, \ldots, n\}$. The poset $NC_n$ is called the lattice of non-crossing partitions.

Find a closed formula for the number of saturated chains from the minimal element $\hat{0}$ to the maximal element $\hat{1}$ in the poset $NC_n$.

Problem 15. Find a bijection between partitions of $n$ with all odd parts and partitions of $n$ with all distinct parts.
Problem 16. Prove that the number of partitions of $n$ with all distinct and odd parts equals the number of self-conjugate partitions of $n$, i.e., partitions $\lambda$ such that $\lambda' = \lambda$. 