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Today we’re going to discuss Eulerian numbers.

Definition 1
Let \( A_{nk} \) be the number of permutations \( w \in S_n \) with exactly \( k \) descents: \( \text{des}(w) = k \).

Notation is sometimes shifted by one index, but this is not that significant.

Fact 2
It’s important not to confuse these with Euler numbers!

There is a nice bijection between permutations in \( S_n \) and increasing binary trees on \( n \) nodes. Remember that one interpretation of Catalan numbers is the number of binary trees! But this time, we’re going to label the vertices by numbers from 1 to \( n \) such that they increase as we go away from the root.

Example 3
Start with \( w = (4, 2, 8, 5, 1, 3, 9, 10, 6, 7) \).

At the root of this tree, we use 1, and the left branch of the tree contains \( (4, 2, 8, 5) \) while the right branch contains everything else. Repeat this process by finding the minimal element in each group, and partition from there!

So each node has at most 1 left child (the smallest element on its left) and at most 1 right child (the smallest element on its right), which means it is indeed a binary tree. The nodes are labeled 1 through \( n \), and they indeed increase downwards.

Fact 4
This is a bijection, since we can just reverse the process by placing 1 in our permutation, putting the left elements to its left and right elements to its right, and so on.

How does this help us say things about Eulerian numbers?
Theorem 5

$A_{nk}$ is the number of increasing binary trees on $n$ nodes with $k$ left edges.

Proof. Think about the bijection. When we go from the permutation to the tree $w \mapsto T_w$, any left edge $i \rightarrow j$ means we have

$$w = (\cdots j \cdots)i \cdots,$$

where $j$ is the minimal element in the parentheses. So $i$ will have a descent! This means that the descents correspond to the vertices with a left child, as desired. \qed

In a sense, Eulerian numbers look a lot like binomial coefficients.

Definition 6

Construct the Eulerian triangle so that the $b$th entry in the $a$th row is $A_{a,b-1}$. The first few rows look like

```
1
1 1
1 4 1
1 11 11 1
1 26 66 26 1
```

This is a lot like Pascal’s triangle, but the weights are different. If we go along the $k$th diagonal from either direction, the weight is $k!$. For example, $26 = 4 \cdot 1 + 2 \cdot 11$.

Theorem 7 (Recurrence relation for Eulerian numbers)

For all $n, k$,

$$A_{n+1,k} = (n - k + 1)A_{n,k-1} + (k + 1)A_{n,k}.$$  

This is not very hard to prove in terms of binary trees or the original descent definition.

Proof. Suppose we have a binary tree with $n$ nodes: we want to add an extra node. We can add an extra leaf with label $n + 1$: it can be checked that the number of ways to add a left edge is $n - k + 1$, and the number of ways to add a right edge is $k + 1$.

Alternatively, we can see this by looking at the permutations: if we have a permutation $w \in S_n$ and we want to add $n + 1$. There are $n + 1$ ways to add it: adding it in one of the $k$ descent slots or at the beginning does not add a descent, and $k + 1$ of these work, while adding it in an ascent slot or at the end adds a descent, which is the other $k - n + 1$ ways. \qed

Remember that last time, we also discussed the two kinds of Stirling numbers. $c(n, k)$, the signless Stirling numbers of the first kind, is the number of permutations $w \in S_n$ with $k$ cycles (including fixed points). Make a Pascal’s triangle such that the $b$th entry in the $a$th row is $c(a, b)$. Then this triangle looks like
Notice that we no longer have the symmetry between $k$ cycles and $n-k$ cycles. But we still have a recurrence relation with weights!

**Proposition 8 (Recurrence relation for $c(n, k)$)**

For all $n, k$,

$$c(n + 1, k) = c(n, k - 1) + nc(n, k).$$

Meanwhile, $S(n, k)$, the Stirling numbers of the second kind, are the number of set-partitions of $[n]$ into $k$ groups. If we construct a similar triangle, it looks like

```
   1
  1  1
 1  3  1
1  7  6  1
1 15 25 10 1
```

It is a bit harder to describe the weights here: they are 1 for all edges sloping down and to the right, and they increase starting from 1 in each row for edges sloping down and to the left.

**Proposition 9 (Recurrence relation for $S(n, k)$)**

For all $n, k$,

$$S(n + 1, k) = S(n, k - 1) + kS(n, k).$$

As an exercise, we should prove these two relations!

There are many more permutation statistics, but we’ll move on to the next topic for now: posets and lattices!

**Definition 10**

A lattice $L$ is a special kind of poset: it has two binary operations **meet** $\land$ and **join** $\lor$. $x \land y$ is the unique maximal element of $L$ which is less than or equal to both $x$ and $y$, and $x \lor y$ is the unique minimal element of $L$ such that it is greater than or equal to both $x$ and $y$.

Here’s an example of a lattice:
However, here’s an example of something that is not a lattice, since \( x \) and \( y \) have no \( x \land y \) and two different potential \( x \lor y \)s:

The usual definition doesn’t actually use posets though: it’s more abstract.

**Definition 11 (Axiomatic definition of a lattice)**

A **lattice** is a set \( L \) with binary operations \( \land \) and \( \lor \) such that

- \( \land \) and \( \lor \) are commutative and associative, so \( x \land y = y \land x \), and \( x \lor y = y \lor x \). Also, \( x \land (y \land z) = (x \land y) \land z \), and similar for \( \lor \).
- \( x \land x = x \lor x = x \).
- (Absorption law) \( x \land (x \lor y) = x = x \lor (x \land y) \).
- \( x \land y = x \) if and only if \( x \lor y = y \).

Note that addition and multiplication do not follow these laws.

**Fact 12**

Given the axiomatic operations and a set \( L \), we can reconstruct a poset, and given any lattice, it and its operations \( \land \) and \( \lor \) satisfy the axioms! In particular, \( x \leq y \) happens if and only if \( x \land y = x \), which is the same as \( x \lor y = y \).

Some time in the future, we will discuss boolean lattices, partition lattices, and Young lattices!