February 15, 2019

Recall the Schensted correspondence between permutations \( w \in S_n \) and pairs of Young tableaux \((P, Q)\): for example, we found that \( w = (3, 5, 2, 4, 7, 1, 6) \) corresponds to

\[
P = \begin{array}{ccc}
1 & 4 & 6 \\
2 & 5 & 7 \\
3 & &
\end{array} \quad Q = \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & 7 \\
6 & &
\end{array}
\]

The process we took to get to \( P \) is

\[
3 \rightarrow 3 \quad 5 \rightarrow 2 \quad 5 \rightarrow 2 \quad 4 \rightarrow 2 \quad 4 \quad 7 \rightarrow 1 \quad 4 \quad 7 \rightarrow P.
\]

According to Schensted’s theorem, the shape \( \lambda = (3, 3, 1) \) tells how many boxes are in each row, and \( \lambda_1 \), the number of boxes in the first row, is the size of the longest increasing subsequence in \( w \). Similarly, the shape \( \lambda' = (3, 2, 2) \) tells us how many boxes are in each column, and \( \lambda'_1 \), the number of boxes in the first column, is the size of the longest decreasing subsequence in \( w \).

We’re going to prove the first half of this (increasing subsequence) and leave the other half as an exercise!

**Definition 1**

The jth basic subsequence in a permutation \( w \), where \( 1 \leq j \leq \lambda_1 \), consists of all entries of \( w \) that were originally inserted in the \( j \)th row.

For example, we have 3 basic subsequences for the example above. For \( B_1 \), note that we inserted 3, 2, and 1, so \( B_1 = (3, 2, 1) \). Similarly, \( B_2 = (5, 4) \), and \( B_3 = (7, 6) \).

**Lemma 2**

Each \( B_j \) is a decreasing sequence.

**Proof.** By construction, if something bumps \( N \), only smaller numbers can do this. So any number after \( N \) must be smaller. \( \square \)
Lemma 3
For all $j \geq 2$, given any $x \in B_j$, we can find $y \in B_{j-1}$ such that $y < x$ and $y$ is located to the left of $x$ in the permutation $w$.

Proof. At the moment of insertion of $x$, take $y$ to be the entry that is located to the left of $x$. It will be less than $x$, and it was already inserted, so it is appears before $x$ in the permutation. □

So now it’s time to prove the Schensted theorem (part 1). $\lambda_1$ is the number of basic subsequences by definition. Note that given any increasing subsequence $x_1 < x_2 < \cdots < x_r$, we can only have at most 1 entry from each $B_j$. So this means $r \leq \lambda_1$. To construct an example of the equality case, pick the last basic subsequence $B_{\lambda_1}$, and pick any $x_{\lambda_1} \in B_{\lambda_1}$. By lemma 2, and we get an $x_{\lambda_1-1} \in B_{\lambda_1-1}$, and so on. Eventually we’ll be done and have an increasing subsequence of length $\lambda_1$!

For the rest of today, we’re going to prove the Hook Length Formula. Recall the theorem:

Theorem 4 (Hook Length Formula)
The number of ways to fill out a standard Young tableau with $|\lambda| = n$ is

$$f^\lambda = \frac{n!}{H(\lambda)} = \frac{n!}{\prod_x h(x)}$$

where $h(x)$ is the “hook length” of $x$. For example, in the below diagram, the hook length $h(x) = 6$.

The original proof was pretty complicated, and it follows from some other formulas. But later, simpler proofs were found, and we’re going to use a random process!

Hook walk proof by Greene, Nijenhuis, Wilf (1979). There is a recurrence relation for the number of Standard Young Tableau. Given a tableau with $n$ boxes, $n$ must appear in one of the bottom-right corner boxes, and removing this corner, we get a standard Young tableau of size $n - 1$. For instance, look at the following:

Since the O’s are the spots that the largest number can be in,

$$f^\lambda = f^{441} + f^{5431} + f^{544}.$$ 

So we’ll try to prove the theorem by induction:

$$f^\lambda = \sum_{v \text{ corner of } \lambda} f^{\lambda-v}.$$ 

Base case: it’s easy to prove this for $n = 0$ or 1.
Inductive step: Now we need to show that the same recurrence relation holds for the expression on the right hand side! In other words, we want to show that

$$ n! \prod_{v \text{ corner of } x} \left( \frac{(n-1)!}{H(\lambda - v)} \right) $$

This is the same as wanting to show that (dividing through by the left hand side)

$$ 1 = \sum_{v \text{ corner}} \frac{1}{n H(\lambda - v)}. $$

We have a 1 on the left side, and we have a bunch of nonnegative integers on the right side. So we can think of this as a probability distribution! We want to construct a random process so that these are probabilities.

Pick any box of $\lambda$ uniformly at random; call it $u$. At each step, we can jump from $u$ to any other square in the hook of $u$ with equal probability. Repeat this process repeatedly, and stop once we reach a corner $v$.

Define $p(v)$ to be the probability that a hook-walk does end at corner $v$. We claim that the probability

$$ p(v) = n H(\lambda - v), $$

which means we would be done!

Why is this? Denote $P(u, v)$ to be the probability that a hook walk $(u, u', u'', \cdots, v)$ starting at box $u$ ends at corner $v$. This is just summing over all hook walks:

$$ P(u, v) = \sum_{u \rightarrow u' \rightarrow \cdots \rightarrow v} P(\text{this hook walk}) = \sum_{u \rightarrow u' \rightarrow \cdots \rightarrow v} \left( \frac{1}{h(u) - 1} \cdot \frac{1}{h(u') - 1} \cdots \right). $$

Here’s the key observation: when we fix $v$, the whole hook-walk stays within the rectangle between the top-left corner and $v$. Whenever we have a rectangle with corners $a, b, c, d$ ($a$ in the top left and $d$ a corner), $h(a) + h(d) = h(b) + h(c)$, and similarly

$$ (h(a) - 1) + (h(d) - 1) = (h(b) - 1) + (h(c) - 1). $$

If $d$ is a corner, though, $h(d) - 1 = 0$, so this simplifies very nicely! The idea is that each hook walk comes with a kind of weight $\frac{1}{h(u) - 1}$, and we have a nice additive identity with inverse weights.

So now consider a rectangular grid of length $k + 1$ by $\ell + 1$. Let’s say the weights of the last row are $\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_k}, B$, and the weights in the last column are $\frac{1}{y_1}, \frac{1}{y_2}, \cdots, \frac{1}{y_{\ell}}, B$.

**Proposition 5**

So now if we sum the weights of lattice paths from $A$ to $B$, the sum of weights is

$$ \frac{1}{x_1 x_2 \cdots x_k y_1 \cdots y_\ell}. $$

For example, for a 2 by 2 grid, if the weights are $\frac{1}{x_1 + y_1}, \frac{1}{y_1}, \frac{1}{x_1}$, and 1, then the total weights are

$$ \frac{1}{x_1 + y_1} + \frac{1}{y_1} + \frac{1}{x_1 + y_1} = \frac{1}{x_1 y_1}. $$

This is an exercise, and we’ll show next lecture that this leads to a proof of the Hook Length Formula!
18.212 Algebraic Combinatorics
Spring 2019

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.