9. Fermionic integrals

9.1. Bosons and fermions. In physics there exist two kinds of particles — bosons and fermions. So far we have dealt with bosons only, but many important particles are fermions: e.g., electron, proton, etc. Thus it is important to adapt our techniques to the fermionic case.

In quantum theory, the difference between bosons and fermions is as follows: if the space of states of a single particle is \( \mathcal{H} \) then the space of states of the system of \( k \) such particles is \( S^k \mathcal{H} \) for bosons and \( \Lambda^k \mathcal{H} \) for fermions. In classical theory, this means that the space of states of a bosonic particle is a usual real vector space (or more generally a manifold), while for a fermionic particle it is an odd vector space. Mathematically “odd” means that the ring of smooth functions on this space (i.e. the ring of classical observables) is an exterior algebra (unlike the case of a usual, even space, for which the ring of polynomial functions is a symmetric algebra).

More generally, one may consider systems of classical particles or fields some of which are bosonic and some fermionic. In this case, the space of states will be a supervector space, i.e. the direct sum of an even and an odd space (or, more generally, a supermanifold — a notion we will define below).

When such a theory is quantized using the path integral approach, one has to integrate functions over supermanifolds. Thus, we should learn to integrate over supermanifolds and then generalize to this case our Feynman diagram techniques. This is what we do in this section.

9.2. Supervector spaces. Let \( k \) be a field of characteristic zero. A supervector space (or simply, superspace) over \( k \) is just a \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space: \( V = V_0 \oplus V_1 \). If \( V_0 = k^m \) and \( V_1 = k^m \) then \( V \) is denoted by \( k^{m|m} \). The notions of a homomorphism, direct sum, tensor product, dual space for supervector spaces are defined in the same way as for \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces. In other words, the tensor category of supervector spaces is the same as that of \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces.

However, the notions of a supervector space and a \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space are not the same. The difference is as follows. The category of vector (and hence \( \mathbb{Z}/2\mathbb{Z} \)-graded vector) spaces has an additional symmetry structure, which is the standard isomorphism \( V \otimes W \rightarrow W \otimes V \) (given by \( v \otimes w \rightarrow w \otimes v \)). This isomorphism allows one to define symmetric powers \( S^n V \), exterior powers \( \Lambda^n V \), etc. for supervector spaces, and also a symmetry \( V \otimes W \rightarrow W \otimes V \), but it is defined differently. Namely, \( v \otimes w \) goes to \( (-1)^{mn}w \otimes v \), \( v \in V_m, w \in V_n \) \((m, n \in \{0, 1\})\). In other words, it is the same as usual except that if \( v, w \) are odd then \( v \otimes w \rightarrow -w \otimes v \). As a result, we can define the superspaces \( S^n V \) and \( \Lambda^n V \) for a supervector space, but they are not the same as the symmetric and exterior powers in the usual sense. For example, if \( V \) is purely odd (\( V = V_1 \)), then \( S^n V \) is the exterior \( m \)-th power of \( V \), and \( \Lambda^n V \) is the \( m \)-th symmetric power of \( V \) (purely even for even \( m \) and purely odd for odd \( m \)).

For a superspace \( V \), let \( \Pi V \) be the same space with opposite parity, i.e. \( (\Pi V)_i = V_{1-i} \), \( i = 0, 1 \). With this notation, the equalities explained in the previous paragraph can be written as: \( S^n V = \Pi^m (\Lambda^n \Pi V) \), \( \Lambda^n V = \Pi^m (S^n \Pi V) \).

Let \( V = V_0 \oplus V_1 \) be a finite dimensional superspace. Define the algebra of polynomial functions on \( V \), \( \mathcal{O}(V) \), to be the algebra \( SV^* \) (where symmetric powers are taken in the super sense). Thus, \( \mathcal{O}(V) = SV_0^* \otimes \Lambda V_1^* \), where \( V_0 \) and \( V_1 \) are regarded as usual spaces. More explicitly, if \( x_1, \ldots, x_n \) are linear coordinates on \( V_0 \), and \( \xi_1, \ldots, \xi_m \) are linear coordinates on \( V_1 \), then \( \mathcal{O}(V) = k[x_1, \ldots, x_n, \xi_1, \ldots, \xi_m] \), with defining relations
\[
x_i x_j = x_j x_i, x_i \xi_r = \xi_r x_i, \xi_i \xi_s = -\xi_s \xi_i
\]
Note that this algebra is itself a (generally, infinite dimensional) supervector space, and is commutative in the supersense. Also, if \( V, W \) are two superspaces, then \( \mathcal{O}(V \oplus W) = \mathcal{O}(V) \otimes \mathcal{O}(W) \), where the tensor product of algebras is understood in the supersense (i.e. \( (a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd) \), where \( p(x) \) is the parity of \( x \)).

9.3. Supermanifolds. Now assume that \( k = \mathbb{R} \). Then by analogy with the above for any supervector space \( V \) we can define the algebra of smooth functions, \( C^\infty(V) := C^\infty(V_0) \otimes \Lambda V_1^* \). In fact, this is a special case of the following more general setting.

**Definition 9.1.** A supermanifold \( M \) is a usual manifold \( M_0 \) with a sheaf \( C^\infty_M \) of \( \mathbb{Z}/2Z \) graded algebras (called the structure sheaf), which is locally isomorphic to \( C^\infty_M \otimes \Lambda(\xi_1, \ldots, \xi_m) \).
The manifold $M_0$ is called the reduced manifold of $M$. The dimension of $M$ is the pair of integers $\dim M_0 | m$.

For example, a super vector space $V$ is a supermanifold of dimension $\dim V_0 | \dim V_1$. Another (more general) example of a supermanifold is a superdomain $U := U_0 \times V_1$, i.e., a domain $U_0 \subset V_0$ together with the sheaf $C^0_U \otimes \Lambda V_1^*$. Moreover, the definition of a supermanifold implies that any supermanifold is “locally isomorphic” to a superdomain.

Let $M$ be a supermanifold. An open set $U$ in $M$ is the supermanifold $(U_0, C^0_M) | (U_0)$, where $U_0$ is an open subset in $M_0$.

By the definition, supermanifolds form a category $\text{SMAN}$. Let us describe explicitly morphisms in this category, i.e. maps $F : M \rightarrow N$ between supermanifolds $M$ and $N$. By the definition, it suffices to assume that $M, N$ are superdomains, with global coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_m$, and $y_1, \ldots, y_p, \eta_1, \ldots, \eta_q$, respectively (here $x_i, y_i$ are even variables, and $\xi_i, \eta_i$ are odd variables). Then the map $F$ is defined by the formulas:

$$y_i = f_{0,i}(x_1, \ldots, x_n) + f_{2,i}^{j_1} (x_1, \ldots, x_n) \xi_{j_1} \xi_{j_2} + \cdots,$$

$$\eta_i = a_{1,i}^j (x_1, \ldots, x_n) \xi_{j_1} + a_{3,i}^{j_1 j_2} (x_1, \ldots, x_n) \xi_{j_1} \xi_{j_2} \xi_{j_3} + \cdots,$$

where $f_{0,i}, f_{2,i}^{j_1}, a_{1,i}^j, a_{3,i}^{j_1 j_2}, \ldots$ are usual smooth functions, and we assume summation over repeated indices. These formulas determine $F$ completely, since for any $g \in C^\infty(N)$ one can find $g \circ F \in C^\infty(M)$ by Taylor’s formula. For example, if $M = N = \mathbb{R}^{1|2}$ and $F(x, \xi_1, \xi_2) = (x + \xi_1 \xi_2, \xi_1, \xi_2)$, and if $g = g(x)$, then $g \circ F(x, \xi_1, \xi_2) = g(x + \xi_1 \xi_2) = g(x) + g'(x) \xi_1 \xi_2$.

**Remark.** For this reason, one may consider only $C^\infty$ (and not $C^r$) functions on supermanifolds. Indeed, if for example $g(x)$ is a $C^r$ function of one variable which is not differentiable $r + 1$ times, then the expression $g(x + \sum_{i=1}^{r+1} \xi_{2i-1} \xi_{2i})$ will not be defined, because the coefficient of $\xi_1 \cdots \xi_{2r+2}$ in this expression should be $g^{(r+1)}(x)$, but this derivative does not exist.

9.4. **Supermanifolds and vector bundles.** Let $M_0$ be a manifold, and $E$ be a vector bundle on $M_0$. Then we can define the supermanifold $M := \text{Tot}(\text{IE})$, the total space of $E$ with changed parity. Namely, the reduced manifold of $M$ is $M_0$, and the structure sheaf $C^\infty_M$ is the sheaf of sections of $\Lambda E^*$. This defines a functor $S : \mathcal{BUN} \rightarrow \text{SMAN}$, from the category of manifolds with vector bundles to the category of supermanifolds. We also have a functor $S_*$ in the opposite direction: namely, $S_*(M)$ is the manifold $M_0$ with the vector bundle $(R/R^2)^*$, where $R$ is the nilpotent radical of $C^\infty_M$.

The following proposition (whose proof we leave as an exercise) gives a classification of supermanifolds.

**Proposition 9.2.** $S_*S = \text{Id}$, and $SS_* = \text{Id}$ on isomorphism classes of objects.

The usefulness of this proposition is limited by the fact that, as one can see from the above description of maps between supermanifolds, $SS_*$ is not the identity on morphisms (e.g. it maps the automorphism $x \rightarrow x + \xi_1 \xi_2$ of $\mathbb{R}^{1|2}$ to $\text{Id}$), and hence, $S$ is not an equivalence of categories. In fact, the category of supermanifolds is not equivalent to the category of manifolds with vector bundles (namely, the category of supermanifolds “has more morphisms”).

**Remark.** The relationship between these two categories is quite similar to the relationship between the categories of (finite dimensional) filtered and graded vector spaces, respectively (namely, for them we also have functors $S, S_*$ with the same properties – check it!). Therefore in supergeometry, it is better to avoid using realizations of supermanifolds as $S(M_0, E)$, similarly to how in linear algebra it is better to avoid choosing a grading on a filtered space.

9.5. **Integration on superdomains.** We would now like to develop integration theory on supermanifolds. Before doing so, let us recall how it is done for usual manifolds. In this case, one proceeds as follows.

1. Define integration of compactly supported (say, smooth) functions on a domain in $\mathbb{R}^n$.
2. Find the transformation formula for the integral under change of coordinates (i.e. discover the factor $|J|$, where $J$ is the Jacobian).
3. Define a density on a manifold to be a quantity which is locally the same as a function, but multiplies by $|J|$ under coordinate change (unlike true functions, which don’t multiply by anything).
Then define integral of compactly supported functions on the manifold using partitions of unity. The independence of the integral on the choices is guaranteed by the change of variable formula and the definition of a density.

We will now realize this program for supermanifolds. We start with defining integration over superdomains.

Let $V = V_0 \oplus V_1$ be a supervector space. The Berezinian of $V$ is the line $\Lambda^{top} V_0 \otimes \Lambda^{top} V_1^*$. Suppose that $V$ is equipped with a nonzero element $dv$ of the Berezinian (called a supervolume element).

Let $U_0$ be an open set in $V_0$, and $f \in C^\infty(U) \otimes \Lambda V_1^*$ be a compactly supported smooth function on the superdomain $U := U_0 \times V_1$ (i.e. $f = \sum f_i \otimes \omega_i$, $f_i \in C^\infty(U)$, $\omega_i \in \Lambda V_1^*$, and $f_i$ are compactly supported). Let $dv_0, dv_1$ be volume forms on $V_0, V_1$ such that $dv = dv_0/dv_1$.

**Definition 9.3.** The integral $\int_U f(v) dv$ is $\int_{U_0}(f(v), (dv_1)^{-1})dv_0$.

It is clear that this quantity depends only on $dv$ and not on $dv_0$ and $dv_1$ separately.

Thus, $\int f(v) dv$ is defined as the integral of the suitably normalized top coefficient of $f$ (expanded with respect to some homogeneous basis of $\Lambda V_1^*$). To write it in coordinates, let $\xi_1, \ldots, \xi_m$ be a linear system of coordinates on $V$ such that $dv = dx_1 \cdots dx_n$ (such coordinate systems will be called unimodular with respect to $dv$). Then $\int f(v) dv$ equals $\int f_{top}(x_1, \ldots, x_n)dx_1 \cdots dx_n$, where $f_{top}$ is the coefficient of $\xi_1 \cdots \xi_n$ in the expansion of $f$.

**9.6. The Berezinian of a matrix.** Now we generalize to the supercase the definition of determinant (since we need to generalize Jacobian, which is a determinant).

Let $R$ be a supercommutative ring. Fix two nonnegative integers $m, n$. Let $A$ be a $n + m$ by $n + m$ matrix over $R$. Split $A$ in the blocks $A_{11}, A_{12}, A_{21}, A_{22}$ so that $A_{11}$ is $n$ by $n$, and $A_{22}$ is $m$ by $m$. Assume that the matrices $A_{11}, A_{22}$ have even elements, while $A_{21}$ and $A_{12}$ have odd elements. Assume also that $A_{22}$ is invertible.

**Definition 9.4.** The Berezinian of $A$ is the element

$$\text{Ber}(A) := \frac{\det(A_{11} - A_{12}A_{22}^{-1}A_{21})}{\det(A_{22})} \in R$$

(where the determinant of the empty matrix is agreed to be 1; so for $m = 0$ one has $\text{Ber}.A = \det A$, and for $n = 0$ one has $\text{Ber}A = (\det A)^{-1}$).

**Remark.** Recall for comparison that if $A$ is purely even then $$\det(A) := \det(A_{11} - A_{12}A_{22}^{-1}A_{21})\det(A_{22}).$$

The Berezinian has the following simpler description. Any matrix $A$ as above admits a unique factorization $A = A_+A_0A_-$, where $A_+, A_0, A_-$ are as above, and in addition $A_+, A_-$ are block upper (respectively, lower) triangular with 1 on the diagonal, while $A_0$ is block diagonal. Then $\text{Ber}(A) = \frac{\det((A_0)_{11})}{\det((A_0)_{22})}$.

**Proposition 9.5.** If $A, B$ be matrices as above, then $\text{Ber}(AB) = \text{Ber}(A)\text{Ber}(B)$.

**Proof.** From the definition using triangular factorization, it is clear that it suffices to consider the case $A = A_-, B = B_+$. Let $X = (A_-)_{21}, Y = (B_+)_{12}$ (matrices with odd elements). Then the required identity is $$\det(1 - Y(1 + XY)^{-1}X) = \det(1 + XY).$$

To prove this relation, let us take the logarithm of both sides and expand using Taylor’s formula. Then the left hand side gives

$$- \sum_{k \geq 1} \text{Tr}(Y(1 + XY)^k(1 - 1kX^{-1}))$$

Using the cyclic property of the trace, we transform this to

$$\sum_{k \geq 1} \text{Tr}((1 + XY)^k)/k$$
(the minus disappears since $X, Y$ have odd elements). Summing the series, we find that the last expression equals
\[ -\text{Tr} \ln(1 - (1 + XY)^{-1}XY) = \text{Tr} \ln(1 + XY), \]
as desired. □

The additive analog of Berezinian is supertrace. Namely, for $A$ as above, $s\text{Tr} A = \text{Tr} A_{11} - \text{Tr} A_{22}$. It is the correct superanalogue of the usual trace, as it satisfies the equation $s\text{Tr}(AB) = s\text{Tr}(BA)$ (while the usual trace does not). The connection between the supertrace and the Berezinian is given by the formula
\[ \text{Ber}(e^A) = e^{s\text{Tr}(A)}. \]

**Exercise.** Prove this formula.

### 9.7. Berezin’s change of variable formula.

Let $V$ be a vector space, $f \in \Lambda V^*$, $v \in V$. Denote by $\frac{\partial f}{\partial v}$ the result of contraction of $f$ with $v$.

Let $U, U'$ be superdomains, and $F : U \to U'$ be a morphism. As explained above, given linear coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_m$ on $U$ and $y_1, \ldots, y_p, \eta_1, \ldots, \eta_q$ on $U'$, we can describe $f$ by expressing $y_i$ and $\eta_j$ as functions of $x_j$ and $\xi_j$. Define the Berezin matrix of $F$, $A := DF(x, \xi)$ by the formulas:
\[ A_{11} = \left( \frac{\partial y_i}{\partial x_j} \right), \quad A_{12} = \left( \frac{\partial y_i}{\partial \xi_j} \right), \quad A_{21} = \left( \frac{\partial \eta_j}{\partial x_j} \right), \quad A_{22} = \left( \frac{\partial \eta_j}{\partial \xi_j} \right). \]

Clearly, this is a superanalogue of the Jacobi matrix.

The main theorem of supercalculus is the following theorem.

**Theorem 9.6.** (Berezin) Let $g$ be a smooth function with compact support on $U'$, and $F : U \to U'$ be an isomorphism. Let $dv, dv'$ be supervolume elements on $U, U'$. Then
\[ \int_{U'} g(v')dv' = \int_U g(F(v))|\text{Ber}DF(v)|dv, \]
where the Berezinian is computed with respect to unimodular coordinate systems.

**Remark.** If $f(\xi) = a +$-terms containing $\xi_j$ then by definition $|f(\xi)| := f(\xi)$ is $a > 0$ and $-f(\xi)$ if $a < 0$.

**Proof.** The chain rule of the usual calculus extends verbatim to supercalculus. Also, we have shown that $\text{Ber}(AB) = \text{Ber}(A)\text{Ber}B$. Therefore, if we know the statement of the theorem for two isomorphisms $F_1 : U_2 \to U_1$ and $F_2 : U_3 \to U_2$, then we know it for the composition $F_1 \circ F_2$.

Let $F(x_1, \ldots, x_n, \xi_1, \ldots, \xi_m) = (x'_1, \ldots, x'_n, \xi'_1, \ldots, \xi'_m)$. From what we just explained it follows that it suffices to consider the following cases.

1. $x'_i$ depend only on $x_j$, $j = 1, \ldots, n$, and $\xi'_i = \xi_i$.
2. $x'_i = x_i + z_i$, where $z_i$ lie in the ideal generated by $\xi_j$, and $\xi'_i = \xi_i$.
3. $x'_i = x_i$.

Indeed, it is clear that any isomorphism $F$ is a composition of isomorphisms of the type 1, 2, 3.

In case 1, the statement of the theorem follows from the usual change of variable formula. Thus it suffices to consider cases 2 and 3.

In case 2, it is sufficient to consider the case when only one coordinate is changed by $F$, i.e. $x'_1 = x_1 + z$, and $x'_i = x_i$ for $i \geq 2$. In this case we have to show that the integral of
\[ g(x_1 + z, x_2, \ldots, x_n, \xi)(1 + \frac{\partial z}{\partial x_1}) - g(x, \xi) \]
is zero. But this follows easily upon expansion in powers of $z$, since all the terms are manifestly total derivatives with respect to $x_1$.

In case 3, we can also assume $\xi'_i = \xi_i$, $i \geq 2$, and a similar (actually, even simpler) argument proves the result. □
9.8. Integration on supermanifolds. Now we will define densities on supermanifolds. Let $M$ be a supermanifold, and \{ $U_\alpha$ \} be an open cover of $M$ together with isomorphisms $f_\alpha : U_\alpha \to U_\alpha'$, where $U_\alpha'$ is a superdomain in $\mathbb{R}^{|n|}$. Let $g_{\beta\beta} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$ be the transition map $f_\alpha f_\beta^{-1}$. Then a density $s$ on $M$ is a choice of an element $s_\alpha \in C^*_M(U_\alpha)$ for each $\alpha$, such that on $U_\alpha \cap U_\beta$ one has

$$s_\beta(z) = s_\alpha(z)\text{Ber}(g_{\alpha\beta})(f_\beta(z)).$$

Remark. It is clear that a density on $M$ is a global section of a certain sheaf on $M$, called the sheaf of densities.

Now, for any (compactly supported) density $\omega$ on $M$, the integral $\int_M \omega$ is well defined. Namely, it is defined as usual calculus: one uses partition of unity $\phi_\alpha$ such that $\text{Supp}\phi_\alpha \subset (U_\alpha)_0$ are compact subsets, and sets $\int_M \omega := \sum_\alpha \int_M \phi_\alpha \omega$ (where the summands can be defined using $f_\alpha$). Berezin’s theorem guarantees then that the final answer will be independent on the choices made.

9.9. Gaussian integrals in an odd space. Now let us generalize to the odd case the theory of Gaussian integrals, which was, in the even case, the basis for the path integral approach to quantum mechanics and field theory.

Recall first the notion of Pfaffian. Let $A$ be a skew-symmetric matrix of even size. Then the determinant of $A$ is the square of a polynomial in the entries of $A$. This polynomial is determined by this condition up to sign. The sign is usually fixed by requiring that the polynomial should be 1 for the direct sum of matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With this convention, this polynomial is called the Pfaffian of $A$ and denoted $\text{PFA}$. The Pfaffian obviously has the property $\text{Pf}(X^TAX) = \text{Pf}(A)\det(X)$ for any matrix $X$.

Let now $V$ be an $2m$-dimensional vector space with a volume element $dv$, and $B$ a skew-symmetric bilinear form on $V$. We define the Pfaffian $\text{Pf}B$ of $B$ to be the Pfaffian of the matrix of $B$ in any unimodular basis by the above transformation formula, it does not depend on the choice of the basis.

It is easy to see (by reducing $B$ to canonical form) that

$$\frac{\wedge^m B}{m!} = \text{Pf}(B)dv.$$

In terms of matrices, this translates into the following (well known) formula for the Pfaffian of a skew symmetric matrix of size $2m$:

$$\text{Pf}(A) = \sum_{\sigma \in \Pi_m} \varepsilon_\sigma \prod_{i \in \{1, \ldots, 2m\}, i < \sigma(i)} a_{i\sigma(i)},$$

where $\Pi_m$ is the set of pairings of $\{1, \ldots, 2m\}$, and $\varepsilon_\sigma$ is the sign of the permutation sending $1, 2, \ldots, 2m$ to $i_1, \sigma(i_1), \ldots, i_m, \sigma(i_m)$ (where $i_r < \sigma(i_r)$ for all $r$). For example, for $m = 2$ (i.e. a 4 by 4 matrix),

$$\text{Pf}(A) = a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}.$$

Now consider an odd vector space $V$ of dimension $2m$ with a volume element $d\xi$. Let $B$ be a symmetric bilinear form on $V$ (i.e. a skew-symmetric form on $\Pi V$). Let $\xi_1, \ldots, \xi_{2m}$ be unimodular linear coordinates on $V$ (i.e. $d\xi = d\xi_1 \wedge \cdots \wedge d\xi_m$). Then if $\xi = (\xi_1, \ldots, \xi_n)$ then $B(\xi, \xi) = \sum_{i,j} b_{ij}\xi_i\xi_j$, where $b_{ij}$ is a skew-symmetric matrix.

Proposition 9.7.

$$\int_V e^{-B(\xi, \xi)}(d\xi)^{-1} = \text{Pf}(B).$$

Proof. The integral equals $\frac{\wedge^m B}{m!} d\xi$, which is $\text{Pf}(B)$. \hfill \Box

Example. Let $V$ be a finite dimensional odd vector space, and $Y = V \oplus V^*$. The space $Y$ has a canonical volume element $dvdv^*$, defined as follows: if $e_1, \ldots, e_m$ be a basis of $V$ and $e_1^*, \ldots, e_n^*$ is the dual basis of $V^*$ then $dvdv^* = e_1 \wedge e_1^* \cdots \wedge e_n \wedge e_n^*$. Let $dy = (dvdv^*)^{-1}$ be the corresponding supervolume element.

Let $A : V \to V$ be a linear operator. Then we can define an even smooth function $S$ on the odd space $Y$ as follows: $S(v, v^*) = (Av, v^*)$. More explicitly, if $\xi_i$ be coordinates on $V$ corresponding to the basis $e_i$, and $\eta_i$ the dual system of coordinates on $V^*$, then

$$S(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m) = \sum a_{ij} \xi_i \eta_j,$$
where \((a_{ij})\) is the matrix of \(A\) in the basis \(e_i\).

**Proposition 9.8.**

\[
\int_Y e^S dy = \det A
\]

**Proof.** We have \(S(v, v_*) = \frac{1}{2}B((v, v_*), (v, v_*))\), where \(B\) is the skew form on \(\Pi Y\), which is given by the formula \(B((v, v_*), (w, w_*)) = (Av, w) - (Aw, v_*)\). It is easy to see that \(\text{Pf}(B) = \det(A)\), so Proposition 9.8 follows from Proposition 9.7.

Another proof can be obtained by direct evaluation of the top coefficient. \(\square\)

9.10. **The Wick formula in the odd case.** Let \(V\) be a \(2m\)-dimensional odd space with a volume form \(d\xi\), and \(B \in S^2V\) a nondegenerate form (symmetric in the supersense and antisymmetric in the usual sense). Let \(\lambda_1, \ldots, \lambda_n\) be linear functions on \(V\) (regarded as the usual space). Then \(\lambda_1, \ldots, \lambda_n\) can be regarded as odd smooth functions on the superspace \(V\).

**Theorem 9.9.**

\[
\int_V \lambda_1(\xi) \cdots \lambda_n(\xi) e^{-\frac{1}{2}B(\xi, \xi)}(d\xi)^{-1} = \text{Pf}(B) \text{Pf}(B^{-1}(\lambda_i, \lambda_j)).
\]

(By definition, this is zero if \(n\) is odd). In other words, we have:

\[
\int_V \lambda_1(\xi) \cdots \lambda_n(\xi) e^{-\frac{1}{2}B(\xi, \xi)}(d\xi)^{-1} = \text{Pf}(B) \sum_{\sigma \in \Pi_m} \varepsilon_{\sigma} \prod_{i \in \{1, \ldots, 2m\}, i < \sigma(i)} (B^{-1}(\lambda_i, \lambda_{\sigma(i)})).
\]

**Proof.** We prove the second formula. Choose a basis \(e_i\) of \(V\) with respect to which the form \(B\) is standard: \(B(e_j, e_l) = 1\) if \(j = 2i - 1, l = 2i\), and \(B(e_j, e_l) = 0\) for other pairs \(j < l\). Since both sides of the formula are polylinear with respect to \(\lambda_1, \ldots, \lambda_n\), it suffices to check it if \(\lambda_1 = e_{i_1}^*, \ldots, \lambda_n = e_{i_n}^*\). This is easily done by direct computation (in the sum on the right hand side, only one term may be nonzero). \(\square\)