3 Example: Inhomogeneity in a small volume

Suppose we are solving 

$$-\nabla \cdot (c \nabla u) = f$$

in $$\Omega = \mathbb{R}^3$$ with a point source $$f(x) = \delta(x - x_0)$$ at $$x_0$$. Furthermore, suppose that $$c(x)$$ is piecewise-constant as in figure 1, with $$c(x) = c_2$$ everywhere except in a volume $$V$$, centered at $$x_1$$, where $$c(x) = c_1$$. Now, suppose that we want the solution $$u(x)$$, but are far from $$V$$: both the source point $$x_0$$ and the desired point $$x$$ are far from $$V$$, with $$|x_1 - x_0|$$ and $$|x_1 - x|$$ both much bigger than the diameter of $$V$$. This is shown schematically in figure 2. In this case, we should expect the effect of the “scattered” solution from $$V$$ to be small at $$x$$, and a Born approximation should apply. Furthermore, we will assume $$c_1 \approx c_2$$ (though not exactly equal!), so that we can neglect the effect of the discontinuity in $$\nabla u$$ mentioned after equation (3) above (which greatly complicates the application of any Born-like approximation in this problem because it would prevent us from using $$u \approx u_0$$ in $$V$$).²

In this case,

$$u_0(x) = G_0(x, x_0)/c(x_0) = \frac{1}{4\pi c_2 |x - x_0|},$$

so in the Born approximation we write:

$$u(x) \approx u_0(x) + \hat{B}u_0,$$

where the scattered part of the solution, applying the SIE form (4) [valid when $$c_1 \approx c_2$$], is

$$\hat{B}u_0 = \ln(c_2/c_1) \iiint_V G_0(x, x') \nabla' u_0(x') \cdot dA'$$

$$= \ln(c_2/c_1) \iiint_V \nabla' \cdot [G_0(x, x') \nabla' u_0(x')] d^3x'$$

$$= \ln(c_2/c_1) \iiint_V [\nabla' G_0(x, x') \cdot \nabla' u_0(x') + G_0 \nabla^2 u_0] d^3x',$$

²It turns out that many people get this wrong in electromagnetism for cases when $$c_1$$ and $$c_2$$ are very different, as discussed in my paper on a closely related subject, “Roughness losses and volume-current methods in photonic-crystal waveguides,” Appl. Phys. B 81, 238–293 (2005): http://math.mit.edu/~stevenj/papers/JohnsonPo05.pdf
where in the second line we applied the divergence theorem, and in the third line the product rule led to a \( \nabla^2 u_0 \) term, where \( \nabla^2 u_0 = -\delta(x - x_0) \) is zero in \( V \) (since \( x_0 \) is outside of \( V \)).

Now, since \( V \) is small compared to the distance from \( x \) and \( x_0 \), the distances \( |x' - x| \) and \( |x' - x_0| \) hardly change for any \( x' \in V \), and so the \( \nabla' G_0 \) and \( \nabla'u_0 \) terms are approximately constant in this integral and we can just pull them out, giving the approximation:

\[
\hat{B}u_0 \approx \ln(c_2/c_1) \nabla' G_0(x, x') \cdot \nabla'u_0(x') \big|_{x'=x_1} \text{ volume}(V).
\]

We can compute these gradients explicitly:

\[
\nabla' \frac{1}{|x' - y|} = -\frac{x' - y}{|x' - y|^3},
\]

and hence:

\[
u(x) \approx \frac{1}{4\pi c_2 |x - x_0|} + \ln(c_2/c_1) \frac{(x_1 - x)}{4\pi |x_1 - x|^3} \cdot \frac{(x_1 - x_0)}{4\pi c_2 |x_1 - x_0|^3} \text{ volume}(V).
\]

Notice that the amplitude of the scattered term vanishes as \( \text{volume}(V) \to 0 \), as expected. Notice that it also depends on the sign of \( (x_1 - x) \cdot (x_1 - x_0) \). Why is that? What does a \( \nabla' G_0 \) source “mean,” physically?

### 3.1 Dipole sources

Consider the following problem in \( \Omega = \mathbb{R}^3 \), requiring as usual that solutions vanish at \( \infty \):

\[
-\nabla^2 D_p(x, x') = -p \cdot \nabla \delta(x - x') = +p \cdot \nabla' \delta(x - x').
\]

This is like the Green’s function equation, except now we have put the derivative of a delta function on the right-hand side, with some constant vector \( p \) (the “dipole moment”). Recall what the derivative of a delta function is:

\[
[-p \cdot \nabla \delta(x - x')][\phi] = [\delta(x - x')][p \cdot \nabla \phi] = p \cdot \nabla \phi \big|_{x'} = \lim_{\epsilon \to 0} \frac{\phi(x + \epsilon p) - \phi(x' + \epsilon p)}{2 \epsilon},
\]

and hence (similar to pset 5 of 2010 or pset 7 of 2011),

\[
-p \cdot \nabla \delta(x - x') = \lim_{\epsilon \to 0} \frac{\delta(x - x' - \epsilon p) - \delta(x - x' + \epsilon p)}{2 \epsilon}.
\]

That is, the derivative of a delta function is a limit of limit of two delta functions of opposite sign, displaced proportional to \( p \). In 8.02, where delta functions are “point charges,” this is what you would have called an “electric dipole.”

We can solve for \( D_p \) quite easily, because we know the solution \( G_0 \) to \( -\nabla^2 G_0(x, x') = \delta(x - x') \), and \( \nabla \) and \( \nabla' \) derivatives can be interchanged in their order:

\[
-p \cdot \nabla \delta(x - x') = p \cdot \nabla' \left[ \delta(x - x') \right] = p \cdot \nabla' \left[ -\nabla^2 G_0(x, x') \right] = -\nabla^2 \left[ p \cdot \nabla' G_0(x, x') \right],
\]

and hence

\[
D_p(x, x') = p \cdot \nabla' G_0(x, x') = p \cdot \frac{x - x'}{4\pi |x - x'|^3}.
\]
In electrostatics, this would be the potential of a dipole. Note that this falls off as \( \sim \frac{1}{|x - x'|^2} \), whereas \( G_0 \) falls off as \( \sim \frac{1}{|x - x'|} \).

Given this solution, we can now interpret the scattered part of the solution (5) above: a small inhomogeneity gives an effective dipole source \( p \) at \( x_1 \), where

\[
p = -\ln\left(\frac{c_2}{c_1}\right) \frac{(x_1 - x_0)}{4\pi|x_1 - x_0|^3} \text{volume}(V).
\]

In electrostatics, for a typical case where \( V \) is a small piece of matter in vacuum, \( c_2 < c_1 \), so \( p \) is parallel to \( x_1 - x_0 \). Physically, a positive point charge induces a dipole moment \( p \) pointed away from the charge, because a “+” charge at \( x_0 \) pushes “+” charges in \( V \) away from it, as shown below.