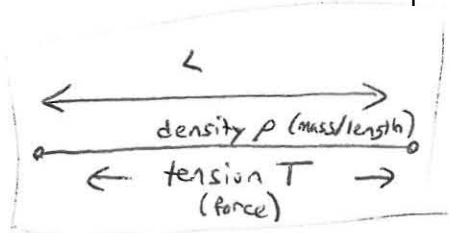


18.303 and Music



* stretched string (piano, guitar, violin, ...):

⇒ scalar wave equation
(neglecting friction, dispersion, ...)

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2}, \quad u(x, t) = \text{displacement}$$
$$u|_{x=0} = 0$$

⇒ eigenfunctions of $\hat{A} = \frac{T}{\rho} \frac{\partial^2}{\partial x^2}$
(= $\hat{A}^* < 0$)

are $u_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ $n=1, 2, \dots$
 $\lambda_n = -\frac{T}{\rho} \left(\frac{n\pi}{L}\right)^2$

⇒ solutions $u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t) \right]$
where $\omega_n = \sqrt{\frac{T}{\rho}} \frac{n\pi}{L}$, α_n from $u(x, 0)$
 β_n from $\dot{u}(x, 0)$

= sum of "normal modes"

$n=1$: fundamental frequency

$n>1$: harmonics

$\omega_1 = \sqrt{\frac{T}{\rho}} \frac{\pi}{L}$
"the note"
"the pitch"

higher tension
and/or
lighter string
and/or
shorter string

⇒ higher pitch

e.g. piano \approx 7 octaves
= 2^7 factor of ω
(highest ω /lowest)
⇒ at same T, ρ
the lowest string would be 128x
longer than highest

* Timbre : the "same" note sounds different on different instruments ... or even on same instrument played differently!

= different amplitudes $\left(\frac{\alpha_n^2 + \beta_n^2}{\alpha_0^2 + \beta_0^2} \right)$ of harmonics ($n > 1$)
 (e.g. due to different initial conditions)

... plus other effects (not captured in simplified scalar wave equation):

- decay rates (depending on n)
- "harmonics" ω_n are not exactly integer multiples (due to dispersion and other effects)
- ...

* Western scales: pitch relationships (intervals) = frequency ratios

- octave = factor of 2 in ω = "same note" because harmonics exactly align
- transposition = multiply all ω 's by same factor \Rightarrow same intervals
- subdivide octave into 12 intervals = semitones (half-steps)

incompatible goals:

- ① modern piano, guitar, ... intervals invariant under transposition \Rightarrow equal ratios \Rightarrow semitone = $2^{1/12}$ ratio "equal temperament"
- ② violin, voice, ... tuned by ear small rational ratios \Rightarrow "nice" sounding intervals \Rightarrow various tunings, "Pythagorean comma", "wolf" intervals \Rightarrow "fifth" = $2^{7/12} = 1.498...$ $\neq \frac{3}{2}$ "just" fifth

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