1. Solve approximately the following two equations

(a) $\epsilon y'' + (1 + x)^2 y' + y = 0$, $0 < x < 1$, with $y(0) = 0$ and $y(1) = 2$.

(b) $\epsilon y'' - (1 + x^2) y' + y = 0$, $0 < x < 1$, with $y(0) = 0$ and $y(1) = 2$

Solutions:

1. Solve approximately the following two equations

In both problems, the highest derivative is multiplied by a perturbation parameter $\epsilon$, and hence the problems are solved using boundary layer techniques.

(a) Since $a(x) = (1 + x)^2 > 0$, the rapidly varying solution is decreasing and thus the boundary layer is at $x = 0$, and it has width $\epsilon$.

We can find the solution outside the boundary layer by solving

$$(1 + x)^2 y'_{\text{out}} + y_{\text{out}} = 0$$

which gives

$$y_{\text{out}} = C_1 e^{\frac{1}{1+x}}$$

where $C_1$ is a constant. Since the rapidly varying function is negligible at $x = 1$, we can use the boundary condition at $x = 0$ to determine the value of $C_1$:

$$y_{\text{out}}(1) = C_1 e^{1/2} = 2$$

thus $C_1 = 2e^{-1/2}$. To find the behaviour of the rapidly varying function near $x = 0$, we look at

$\epsilon y'' + (1 + 0)^2 y' = 0$

which gives

$$y_r \approx C_2 e^{-x/\epsilon}$$

Using the boundary condition at $x = 0$, i.e

$$C_2 + y_{\text{out}}(0) = 0$$

we find $C_2 = -2e^{-1/2}$. Hence the solution is

$$y_{\text{uniform}} = 2e^{-1/2} e^{\frac{1}{1+x}} - 2e^{-1/2} e^{-x/\epsilon}$$

1. (b) Since $a(x) = -(1 + x^2) < 0$, the rapidly varying solution is increasing and thus the boundary layer is at $x = 1$, and it has width $\epsilon$. 

We can find the solution outside the boundary layer by solving
\[-(1 + x^2)y''_\text{out} + y_\text{out} = 0\]
which gives
\[y_\text{out} = C_1 e^{\tan^{-1} x}\]
where \(C_1\) is a constant. Since the rapidly varying function is negligible at \(x = 0\), we can use the boundary condition at \(x = 0\) to determine the value of \(C_1\):
\[y_\text{out}(0) = C_1 e^0 = 0\]
thus \(C_1 = 0\). Hence \(y_\text{out} \equiv 0\). Therefore the solution is approximately zero outside the boundary layer.

To find the behaviour of the rapidly varying function near \(x = 1\), we look at
\[\epsilon y'' - (1 + 1^2)y' = 0\]
which gives
\[y_r \approx C_2 e^{2(x-1)/\epsilon}\]
Using the boundary condition at \(x = 1\), we find \(C_2 = 2\). Hence one may think that the solution can be approximated by
\[y_{\text{uniform}} = y_r + y_\text{out} = y_r = 2e^{2(x-1)/\epsilon}\]  
(1)
That is correct in that the absolute error between the exact solution and \(y_{\text{uniform}}\) given by (1) will be small. But since the solution itself is also ”small”, the relative error may be large. In such a case, it may be useful to calculate higher order approximations to \(y_r\) which, in the present case, equals \(y_{\text{uniform}}\).

To calculate a better approximation, we let
\[y_r = e^{\frac{1}{\epsilon} \int_1^x (1 + x^2) dx} v(x) = e^{\frac{1}{\epsilon} (x + \frac{x^3}{3} - \frac{2}{3})} v(x)\]
where \(v(x)\) is expected not to be rapidly varying. Substituting this into the differential equation, we obtain
\[v' = \frac{2x + 1}{1 + x^2} v = \epsilon v''\]
Neglecting the right hand side, we find
\[\ln v = -\ln(1 + x^2) - \tan^{-1} x + c\]
or
\[v(x) = \frac{2C}{1 + x^2} e^{-\left(\tan^{-1} x - \frac{x}{2}\right)}\]
where \(c\) and \(C\) are some constants. Using the boundary condition at \(x = 1\), we see that \(C = 2\). Hence we find
\[y_{\text{uniform}} = y_\text{out} + y_r = y_r = \frac{4}{1 + x^2} e^{-\left(\tan^{-1} x - \frac{x}{2}\right)} e^{\frac{1}{\epsilon} (x + \frac{x^3}{3} - \frac{2}{3})} dx\]
which is more accurate than (1).