Problem: Consider the differential equation
\[ y'' + x^4 y = 0. \]

(a) Locate and classify the singular points, finite or infinite, of this differential equation. (10%)
(b) Find the WKB solutions of this equation. For what values of \( x \) are the WKB solutions good approximations of the solution? (25%)
(c) Find the Maclaurin series solution of this equation. For what values of \( x \) are these series convergent? (25%)
(d) Find the entire asymptotic series of the solutions which are useful when the magnitude of \( x \) is very large. For what values are these series convergent? (25%)
(e) Find the exact solution of this equation. (15%)

Solutions:
(a) and (b):
The infinity is an irregular singular point of this equation.
Since \( x^4 \) is always positive, the WKB solutions are oscillatory. Putting \( p = x^2 \), \( \int pdx = x^3/3 \),
we have\[ y_WKB = x^{-1} \exp(\pm ix^3/3). \]
The rank of the irregular singular point at \( \infty \) is 3.

(c).
Let \( y = \sum a_n x^n \), \( a_{-1} = a_{-2} = \ldots = 0. \)
We have \( y'' = \sum a_n n(n - 1)x^{n-2} \)
and \( x^4 y = \sum a_n x^{n+4} = \sum a_{n-6} x^{n-2}. \)
Thus the recurrence formula is \( a_n n(n - 1) = -a_{n-6}. \)
Let \( n = 6m, \)
then
\[ a_{6m} = -\frac{a_{6(m-1)}}{6^2 m(m - 1/6)} = (-1)^m \frac{a_0 \Gamma(5/6)}{6^{2m} m! \Gamma(m + 5/6)}. \]
Thus one of the Maclaurin solutions is
\[ y_1 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{6m}}{6^{2m} m! \Gamma(m + 5/6)}. \]
Setting \( n = 6m + 1, \)
we have
\[ a_{6m+1} = -\frac{a_{6(m-1)+1}}{6^2 (m + 1/6)(m)} = (-1)^m \frac{a_1 \Gamma(7/6)}{6^{2m} m! \Gamma(m + 7/6)}. \]
Thus the second Maclaurin solution is
\[ y^2 = \sum_{m=0}^{\infty} \frac{(-1)^m x^{6m+1}}{6^{2m} m! \Gamma(m + 7/6)}. \]

These series converge for all finite values of \( x \).

(d) Let
\[ y = \exp(ix^3/3)Y, \]
then
\[ (D + ix^2)(D + ix^2)Y + x^4Y = 0, \]
or
\[ Y'' + 2ix^2Y' + 2ixY = 0. \]
Let
\[ Y = \sum A_n x^{-n}, \quad A_{-1} = A_{-2} = \cdots = 0. \]
We have
\[ Y'' = \sum A_n (n+1)(n+2)x^{-3-n} = \sum A_{n-3}(n-2)(n-1)x^{-n} \]
and
\[ 2ix^2Y' + 2ixY = -2i \sum A_n nx^{-n}. \]
We get
\[ A_n = \frac{(n-2)(n-1)}{2in} A_{n-3}, \quad n > 0. \]
Setting \( n = 3m \), we get
\[ A_{3m} = \frac{3(m - 2/3)(m - 1/3)}{2im} A_{3(m-1)} = \left( \frac{3}{2t} \right)^m \frac{\Gamma(m + 1/3) \Gamma(m + 2/3)}{m! \Gamma(2/3) \Gamma(1/3)}. \]
Thus one of the asymptotic solutions is
\[ y_3 = e^{ix^3/3} \sum_{m=0}^{\infty} \frac{(3/2t)^m \Gamma(m + 1/3) \Gamma(m + 2/3)x^{-3m-1}}{m!}. \]
The second asymptotic solution is obtained by taking the complex conjugate of \( y_3 \). We get
\[ y_4 = e^{-ix^3/3} \sum_{m=0}^{\infty} \frac{(3i/2)^m \Gamma(m + 1/3) \Gamma(m + 2/3)x^{-3m-1}}{m!}. \]
These series converge for no value of \( x \).

(e) The asymptotic forms of the Bessel functions are \( t^{-1/2} \exp(\pm it) \).
Thus we put
\[ x^3/3 = t, \quad y = t^{1/6}Z. \]
We have
\[ \frac{d}{dx} = 3^{2/3} t^{2/3} \frac{d}{dt}. \]
Therefore
\[ \frac{d^2y}{dx^2} = 3^{4/3} t^{2/3} \frac{d}{dt} \left( \frac{d}{dt} t^{1/6}Z \right) = 3^{4/3} t^{3/2} \left( \frac{d}{dt} + \frac{5}{6t} \right) \left( \frac{d}{dt} + \frac{1}{6t} \right) Z, \]
and the differential equation for \( Z \) is
\[ \frac{d^2Z}{dt^2} + \frac{1}{t} \frac{dZ}{dt} + (1 - \frac{1}{36t^2})Z = 0. \]
The solution of the equation above is,
\[ Z = aJ_{1/6}(t) + bJ_{-1/6}(t). \]
Hence
\[ y(x) = ax^{1/2}J_{1/6} \left( \frac{x^3}{3} \right) + bx^{1/2}J_{-1/6} \left( \frac{x^3}{3} \right). \]