18.306 Advanced Partial Differential Equations with Applications
Fall 2009

Problem Set Number 03

18.306 — MIT (Fall 2009)

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1.1 Statement: Linear 1st order PDE (problem 10) T.L.R.

Do the **Task Left to the Reader** (T.L.R.) at the end of the answer to the problem *Linear 1st order PDE (problem 10)* in Problem set # 2 (this is in p.p. 14-15 of the Answers to Probelm Set Number

1.2   Statement: Riemann Problems (problem 01).

Consider the following conservation law (in a-dimensional variables)

$$u_t + \left(\frac{1}{2}u^2\right)_x = 1, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0, \qquad (1.1)$$

where u is conserved, and shocks are used to avoid multiple-valued solutions.

Find the solution to the Riemann problem for this equation. Namely, for the initial values

$$u(x, 0) = a \quad \text{for} \quad x < 0 \quad \text{and} \quad u(x, 0) = b \quad \text{for} \quad x > 0, \qquad (1.2)$$

where a and b are arbitrary real constants $-\infty < a,\ b < \infty$. **Hint:** *The solution involves shocks, expansion fans, and regions where u depends on time only. Expansion fans are regions where all the characteristics emanate from a single point in space time.*

1.2.1   Justification of quadratic fluxes.

Here we justify the use of conservation laws of the form

$$u_t + \left(\frac{1}{2}u^2\right)_x = S, \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0, \qquad (1.3)$$

where S is some source term, u is conserved and can be both positive or negative, and shocks are used to avoid multiple-values in the solution.

Consider a scalar conservation law in 1-D, with a source term,

$$\rho_t + q_x = S, \qquad (1.4)$$

where $\rho = \rho(x, t)$ is the density of some conserved quantity (hence $\rho \geq 0$), $q = Q(\rho)$ is the corresponding flux, and $S = S(\rho, x)$ is the density of sources/sinks. **Assume now that S is "small"** — this is made precise below in item **2**. Then solutions where ρ is close to a constant should be possible. Hence let $\rho_0 > 0$ be some fixed (constant) density value, and proceed as follows:

1. Expand Q near ρ_0 using Taylor's theorem

$$Q = q_0 + c_0 \left(\rho - \rho_0\right) + \frac{s_1}{2\,\rho_0}\left(\rho - \rho_0\right)^2 + \dots, \tag{1.5}$$

where q_0 is the flux for $\rho = \rho_0$, c_0 is the corresponding characteristic speed, and s_1 is a constant with the dimensions of a velocity. We now **assume that $s_1 \neq 0$; in fact, that**[1] $s_1 > 0$. *Notice that s_1 is a measure of how nonlinear the equation in (1.4) is. The further away for zero s_1 is, the stronger the leading order nonlinear term in the equation is.*

2. Let $S_1 > 0$ be some "typical" value for the source term size, and let $L > 0$ be some "typical" length scale. Then **the source term is small in the sense that $0 \ll \epsilon^2 = \dfrac{L\,S_1}{s_1\,\rho_0} \ll 1$.**

Introduce the a-dimensional variables

$$\tilde{x} = \frac{x - c_0\,t}{L} \quad \text{and} \quad \tilde{t} = \frac{\epsilon\,s_1}{L}\,t, \quad \text{with} \quad \rho = \rho_0\left(1 + \epsilon\,u\right) \quad \text{and} \quad S = S_1\,\tilde{S}. \tag{1.6}$$

Then (1.4) becomes

$$u_{\tilde{t}} + \left(\frac{1}{2}\,u^2 + O(\epsilon)\right)_{\tilde{x}} = \tilde{S}. \tag{1.7}$$

Upon neglecting the $O(\epsilon)$ term, this has the form in (1.3).

1.3 Statement: Riemann Problems (problem 02).

Consider the following conservation law (in a-dimensional variables) for the density u

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \delta(x), \quad \text{for} \quad -\infty < x < \infty \quad \text{and} \quad t > 0, \tag{1.8}$$

where shocks are used to avoid multiple-valued solutions, and $\delta(*)$ stands for Dirac's delta function.

Solve the Riemann problem for (1.8) given by the initial data

$$u(x,\,0) = a \quad \text{for} \quad x < 0 \quad \text{and} \quad u(x,\,0) = b \quad \text{for} \quad x > 0, \tag{1.9}$$

where $-\infty < a,\ b < \infty$ are arbitrary real constants. **Hint:** *The solution involves shocks, expansion fans,[2] and regions where u is constant. The information in § 1.3.1 should prove useful.*

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[1]If $s_1 < 0$, a similar analysis is possible.

[2]Expansion fans are regions where all the characteristics emanate from a single point in space time.

1.3.1 What the equation means. Causality.

The delta function (point source term) has meaning via the integral form of the conservation law; namely:

$$\frac{d}{dt} \int_{x_L}^{x_R} u \, dx = 1 + \frac{1}{2} u^2(x_L, \, t) - \frac{1}{2} u^2(x_R, \, t), \tag{1.10}$$

for **any** constants $x_L < 0 < x_R$. Hence the solutions to (1.8) are "regular" solutions of the conservation law

$$u_t + \left(\frac{1}{2} u^2 \right)_x = 0, \tag{1.11}$$

away from the position $x = 0$ of the point source, and have a **discontinuity at $x = 0$ satisfying**

$$u_R^2 - u_L^2 = 2, \tag{1.12}$$

where u_L (resp. u_R) is the value of the solution on the left (resp. right) side of the discontinuity. In addition, **the discontinuity at $x = 0$ should satisfy causality:**

> *Every point in space time should be connected, via a single characteristic, to a point in the past where data is prescribed.* (1.13)

This restricts the solutions to (1.12). **Not all pairs $(u_L, \, u_R)$ satisfying (1.12) are acceptable for the solutions to (1.8).** Below we show that <u>the allowed pairs at $x = 0$ are:</u>

P1. $0 < u_L < u_R = \sqrt{u_L^2 + 2}$. The characteristics enter the discontinuity from the left, and exit on the right with a larger value for u. See figure 1.1.

P2. $u_R = -\sqrt{u_L^2 + 2} < 0 < u_L$. The characteristics on both sides of the discontinuity enter it. As shown below, this corresponds to a shock coalescing with the discontinuity. See figure 1.1.

P3. $0 = u_L < u_R = \sqrt{2}$. The characteristics on the left of the discontinuity are parallel to it. Characteristics carrying the value $u = \sqrt{2}$ exit the discontinuity on the right. See figure 1.1.

P4. $u_R = -\sqrt{2} < u_L = 0$. The characteristics enter the discontinuity from the right, and do not come out. On the left the characteristics are parallel to the discontinuity. See figure 1.2.

P5. $u_R = -\sqrt{u_L^2 + 2} < u_L < 0$. The characteristics enter the discontinuity from the right, and exit on the left with a larger value for u. See figure 1.2.
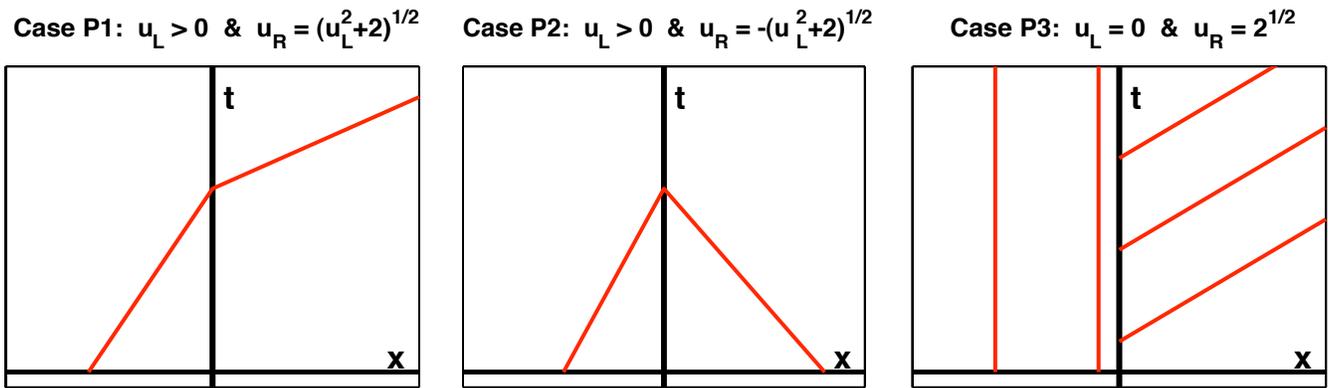
Figure 1.1: Plot of a few typical characteristic curves for the cases in items **P1 − P3.**

The excluded pairs at $x = 0$ are:

Ex. $u_L < 0 < u_R = \sqrt{u_L^2 + 2}$. Characteristics emanate from the discontinuity towards both the left and right sides. See figure 1.2.
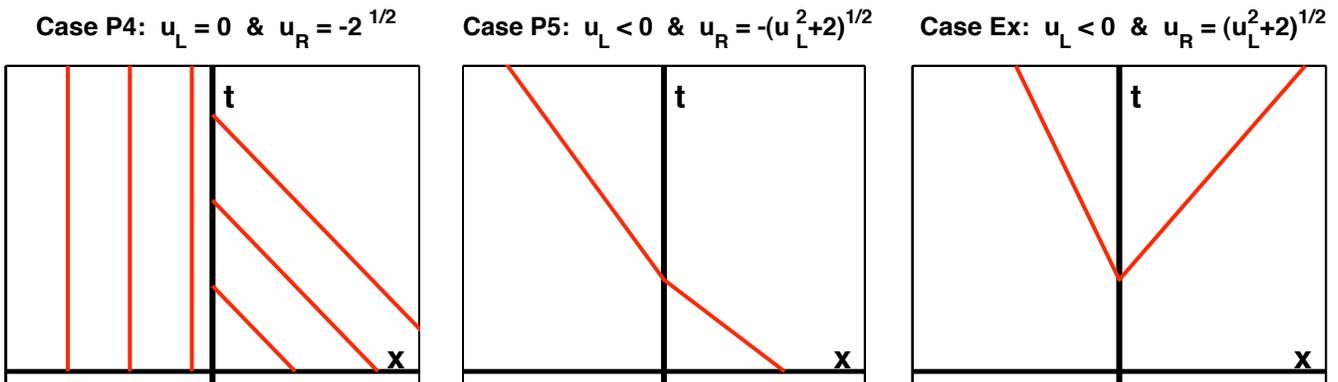
Figure 1.2: Plot of a few typical characteristic curves for the cases in items **P4 − P5** and **Ex.**

Clearly, **P1 − P5** and **Ex** cover all the possible solutions to (1.12).

In order to understand why the cases in **P1 − P5** are allowed, while the case in **Ex** is excluded, it is *convenient to think of equation (1.8) as the limit $\epsilon \downarrow 0$ of the following equation*

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \frac{1}{\epsilon}\,f\left(\frac{x}{\epsilon}\right), \tag{1.14}$$

where $f = f(x)$ is a smooth function with the following properties: (i) $f > 0$ for $-1 < x < 1$. (ii) $f = 0$ for either $x \leq -1$ or $x \geq 1$. (iii) $\int f(x)\,dx = 1$. The advantage of doing this is that, while the characteristic form for (1.8)

$$\frac{du}{dt} = \delta(x) \quad \text{along} \quad \frac{dx}{dt} = u \tag{1.15}$$

does not have a clear meaning at $x = 0$, the characteristic form for (1.14)

$$\frac{du}{dt} = \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) \quad \text{along} \quad \frac{dx}{dt} = u \tag{1.16}$$

makes perfect sense for any $\epsilon > 0$. Hence, we can study what the characteristics do in this case, and then see what happens in the $\epsilon \downarrow 0$ limit. Below we use this approach to show that the cases in **P1 − P5** are allowed, while the case in **Ex** must be excluded.

-1- **Case P1.** Consider characteristics that enter the interval $I_0 = \{x | -\epsilon < x < \epsilon\}$ from the left (necessarily with a value $u = u_L > 0$), and traverse I_0. Clearly, u increases as this happens. Hence, after a while (provided that they are not intersected by a shock) the characteristics exit on the right, with a larger value of $u = u_R$. This is precisely the scenario of case **P1** — note that, in the limit $\epsilon \downarrow 0$, the characteristics cross I_0 in zero time.

The magnitude of the jump in u as the characteristics cross I_0 follows from noticing that (1.16) yields, for $u > 0$,

$$\frac{d}{dx}\left(\frac{1}{2}u^2\right) = \frac{1}{\epsilon} f\left(\frac{x}{\epsilon}\right) \tag{1.17}$$

Upon integrating this from $x = -\epsilon$ to $x = \epsilon$, equation (1.12) follows.

-2- **Case P3.** Consider now characteristics that start somewhere inside the interval I_0, with $u = 0$. In this case both the values of u and x increase with time. Hence, after a while, the characteristics exit on the right, with some value $u_1 > 0$.

Let Δt be the time that a characteristic spends inside I_0. What can we say about Δt? If the characteristic starts close to $x = \epsilon$, then Δt is small. On the other end, if it starts close to $x = -\epsilon$, u and x initially grow very slowly — since $f(x)$ is small near $x = -1$ — resulting in a large Δt. Thus Δt can take any value in the range $0 < \Delta t < \infty$. Furthermore, as $\epsilon \downarrow 0$, the characteristic has to start closer and closer to $x = -\epsilon$ to prevent Δt from vanishing — but the full range $0 < \Delta t < \infty$ persists in the limit.

What can we say about u_1? It is difficult to say what u_1 is when $\epsilon > 0$. However, in the limit $\epsilon \to 0$, if $\Delta t > 0$, integration of (1.17) from $x = -\epsilon$ to $x = \epsilon$, yields $u_1 = \sqrt{2}$.

Thus, the situation in **case P3 can be interpreted as follows:** The value $u = 0$ to the left of the discontinuity at the origin extends all the way to the left "inside" boundary of the discontinuity. Characteristics along the left "inside" boundary of the discontinuity stay there — with zero speed — for some arbitrary time Δt, at which point they experience an infinite acceleration, and exit to the right of the discontinuity with the value $u = \sqrt{2}$. Note that:

$$\text{The causality restriction in (1.13) is NOT violated,} \tag{1.18}$$

even though, at first sight, it would seem as if this case does not satisfy causality, with information being injected from the discontinuity at $x = 0$ into the solution.

-3- **Case P5.** Consider characteristics that enter the interval I_0 from the right (necessarily with a value $u = u_R < 0$), and traverse I_0 right to left. Then u increases as this happens, and the characteristics slow down in its progress towards the left side of I_0. However, if $u = u_R$ is negative enough — $u_R < u_C < 0$, for some critical value u_C — the characteristics make it to the other side,[3] and exit on the left with some (still negative) value $u = u_L$.

Again, using (1.18) one can see that $u_C = -\sqrt{2}$, and that (1.12) applies if the characteristics make it across I_0.

-4- **Case P4.** Consider the same situation as in the prior case, but assume now that the value with which the characteristics enter I_0 from the right is "critical", $u_R = u_C = -\sqrt{2}$. In this case the characteristics slow down as they approach[4] $x = -\epsilon$, with $u \uparrow 0$ as this happens. The characteristics never make it out of I_0, and stay trapped there forever.

At this point, a natural question to ask is: what happens if the characteristics enters I_0 from the right with a value $u_C < u_R < 0$? For the answer, see remark **1.2** below.

-5- **Case P2.** In all the prior cases we considered characteristics entering I_0 from the left, or the right, and proceeding forward in time without encountering characteristics coming the

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[3]Assuming that they do not encounter a shock on the way there.
[4]Assuming that they do not encounter a shock that "terminates" them.

opposite way (thus triggering the need to insert a shock wave). Consider now what happens when characteristics enters I_0 from the left with some value $u = u_L > 0$, and another set enters from the right carrying $u = u_R < 0$. Then a shock will have to occur inside I_0, where both sets of characteristics terminate. However, if this shock does not have zero speed, it will exit to the left or right (in zero time in the limit $\epsilon \downarrow 0$) — changing the value of u on the left (or right) of the discontinuity at $x = 0$ to a value different from u_L (or u_R). Thus this scenario can happen in the $\epsilon \downarrow 0$ limit only if the shock speed is zero. From conservation, this corresponds to the case where u_L and u_R are related by (1.12) — see remark **1.1** below.

-6- **Case Ex.** For this case to apply, the characteristics would have to start inside the interval I_0, travel for a while staying inside — even in the limit $\epsilon \downarrow 0$, and then exit to both left and right. However: (i) u increases while a characteristic stays within I_0. (ii) For a characteristic to stay inside I_0 for a finite amount of time, in the limit $\epsilon \downarrow 0$, it has to start with $u = 0$ — if $u < 0$, then it exits immediately to the left. Hence, a characteristic that travels "inside" the discontinuity for any finite period of time, can only exit to the right. This shows that this case is not possible. This case violates causality, as it requires characteristics to be created "out of nowhere" on the left boundary of the discontinuity.

Remark 1.1 Rankine-Hugoniot conditions for shocks with a point source tracking them.

Consider the situation where a shock appears in a conservation equation for some quantity ρ, and then (by some means) a point source of ρ is added right at the location of the shock. The relevant equation is then

$$\rho_t + q_x = A(t)\,\delta(x - x_s(t)), \tag{1.19}$$

where $A(t)$ is the amount of conserved "stuff" that is deposited at the shock position per unit time, $x = x_s(t)$ is the shock position, and q is the flux. Write now the integral form of this equation

$$\frac{d}{dt} \int_c^d \rho\,dx = q(c,\,t) - q(d,\,t) + A, \tag{1.20}$$

where $c < x_s < d$ are some arbitrary (fixed) points. Then: **(i)** *Write $\int_c^d \rho\,dx = \int_c^{x_s} \rho\,dx + \int_{x_s}^d \rho\,dx$.* **(ii)** *Note that ρ is "nice" on each side of the shock, and satisfies $\rho_t + q_x = 0$.* **(iii)** *Take the time*

derivative of the integral in (1.20) using (i), and the standard rules for taking derivatives of integrals with variable boundaries. **(iv)** *Use (ii) to simplify the result of (iii). This yields:*

$$\frac{dx_s}{dt} = \frac{q_L - q_R}{\rho_L - \rho_R} + \frac{A}{\rho_L - \rho_R},$$

(1.21)

where the subscript L indicates values immediately to the left of the shock, and the subscript R indicates values immediately to the right. Notice that, with $dx_s/dt = 0$, $u = \rho$, $q = (1/2)\,u^2$, and $A = 1$, this reduces to (1.12).

Remark 1.2 *Let us now go back to items* **-3-** *and* **-4-** *above, and ask the question:* What happens if the characteristics enter I_0 from the right, with a value $-\sqrt{2} = \boldsymbol{u_C} < \boldsymbol{u_R} < \boldsymbol{0}$? *Then, somewhere inside I_0 the characteristics reverse direction, start moving to the right, and cross the characteristics (from later times) still moving left. Hence a shock is required. In the limit $\epsilon \downarrow 0$, this shock will exit to the right in zero time, and change the value of the solution to the right of $x = 0$ to some other value — hence* this scenario is NOT allowed in the limit $\epsilon \downarrow 0$.
Question: Why does the shock exit to the right? *Because, if it exits to the left, it will not be able to suppress the crossing of characteristics that it is supposed to prevent (which happens inside I_0). Neither can it stay inside I_0, because, just as in item* **-5-**, *this requires that (1.12) be satisfied for some $u_L > 0$. But there is no such u_L if $-\sqrt{2} < u_R < 0$!*

1.4 Statement: KdV-Burgers Equation (problem 01).

Consider the p.d.e. (in a-dimensional variables)

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \nu\,u_{xx} + \mu\,u_{xxx},$$

(1.22)

where $\nu > 0$ and $\mu \neq 0$ are both small. This p.d.e. has been proposed as a simple model for the structure of bores in shallow water.[5]

Your task is: Study the behavior of the physically meaningful traveling wave solutions of this equation, as a function of the equation parameters ν, μ, and the wave parameters (e.g.: wave amplitude, mean value, speed). Are there solutions that approach constants as $x \pm \infty$? If so, how

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[5]See §13.15 of Whitham's *Linear and nonlinear waves.*

are the constants approached (monotone, oscillations)? Are the constants equal, or different? Are there periodic, oscillatory solutions? Make plots sketching the types of solutions that are possible.

Note: to be physically meaningful, a solution has to be bounded.

HINT. Exploit the symmetries of the problem to reduce the number of free parameters: Because the equation is Galilean invariant, you need only look at time independent solutions (zero propagation speed)

$$\left(\frac{1}{2}\,u^2\right)_x = \nu\,u_{xx} + \mu\,u_{xxx}, \tag{1.23}$$

This can be integrated to

$$\mu\,u_{xx} = -\nu\,u_x + \frac{1}{2}\,u^2 + \kappa, \tag{1.24}$$

where κ is a constant. By a re-scaling of the dependent and independent variables, this last equation can be reduced to one of the following three cases, each having a **single parameter**

$$u_{xx} = -\delta\,u_x + \frac{1}{2}\,u^2 + \sigma, \tag{1.25}$$

where $\sigma = \pm 1$ or $\sigma = 0$, and $0 < \delta < \infty$. Study now the physically relevant solutions of this equation, as δ varies.

Remark 1.3 *There is no explicit solution for (1.25). Reduce it to a phase plane system, find the critical points and their type, the null-clines, etc., and use this information to deduce what the phase portrait for the equation looks like. From this all the information needed to do the problem follows.*

THE END.