Example 1:

Compare behavior, with initial smooth "bump" profile $\rho = f(x)$, or $A = f(x)$ for TRAFFIC FLOW and FLOOD WAVES. List resulting differences in behavior.


c) Characteristics almost always cross.

WRITE THE PRECISE CONDITIONS NEEDED FOR THIS TO HAPPEN. WHEN/WHERE DO CHARACTERISTICS CROSS. FIRST CROSSING.

Simple problem: $\rho_t + q_x = \rho_t + c(\rho)\rho_x = 0,$

$\rho(x, 0) = R(x),$

c(\rho) = dq/d\rho.$

Solution by characteristics: $x = X(s, t) = C(s)\cdot t + s, \ (#1)$

$\rho = R(s),$

where $C(x) = c(R(x)) = \text{wave speed along initial data}.$

Characteristics do not cross if and only if can solve for $s$ as a function of $x$ and $t$ --- $s = S(x, t)$ --- from (#1) if and only if map $s \rightarrow x$ is monotone: $X_s \neq 0.$

That is: inspect $X_s = C'(s)\cdot t + 1.$

So, if $C'(s) < 0$ somewhere, there will be a time when $X_s = 0.$

Thus, the condition for crossing is:

$dC/dx < 0$ somewhere in the initial data.

Graphics: show how $x = X(s, t) = C(s)\cdot t + s$ looks like as a function of $s$, for $t$ fixed, as $t$ grows, if $C(s)$ is a localized hump.

t = 0: straight line $x = s.$

t > 0, moderate: straight line develops a wiggle.

t > 0, large: wiggle large enough to produce a local max. and a local min. Hence a range where map is not 1-to-1.

Formula for critical time $t_c$ and location $x_c$, where the characteristics cross first, assuming $C'(s) < 0$ somewhere (no crossings otherwise):

Let $s_c$ be the value of $s$ at which $C'(s)$ reaches its largest negative value (i.e.: absolute minimum). Then $t_c = -1/C'(s_c)$ ... so $X_s$ vanishes.

$x_c = C(s_c)\cdot t_c + s_c.$

Extras: To (graphically) visualize infinities of $\rho_x$ and $\rho_t$, characterized by the solutions to $1 + t^*C'(s)$, plot $y = C(s)$ versus $y = -1/t$ (horizontal line).