Separation of Variables and Normal Modes

Another way to solve the wave equation (works for other equations too).

Example: do heat equation in 1D with T=0 at ends.
do wave equation in 1D with $u_x = 0$ at ends.

Note: for the wave equation the solutions obtained in this way should be compatible
with the form $u = f(x\cdot c^* t) + g(x+c^* t)$.

Exercise:
Typical separation of variables solution has the form:

$$u = \cos(n^* \pi^* t/L)*\sin(n^* \pi^* x/L)$$

which yields

$$u = \left(1/2\right)*\cos((n^* \pi^*/L)*(x+t)) + \left(1/2\right)*\cos((n^* \pi^*/L)*(x-t))$$

using trigonometric equalities.

Normal modes. Equations of the form $u_t = Lu$
Relationship with separation of variables: equation invariant under time shift allows
separation $u = \exp(\lambda^* t) U(x)$

Example: write wave equation as $u_t = v$ and $v_t = u_{xx}$.
heat equation: $u_t = u_{xx}$

• Note analogy with linear o.d.e. $dY/dt = A*Y$, A NxN matrix, solved by finding
eigenvalues and eigenvectors of A.

• Hence look for solutions of the form $u = e^{\lambda^* t} v(x)$. Solve and find normal
modes (eigenvalues and eigenfunctions).

General solution: Superposition ... leads to Fourier Series, etc.

Example: heat equation with various B.C.
1. In a ring: periodic.
2. Zero T at ends.
3. Zero flux at ends.

Various types of Fourier series.

Explain this works, for example, as long as the associated eigenvalue problem is self-
adjoint:
1) Interpretation of matrices as linear operators.
2) Interpretation of self-adjoint for matrices in terms of the scalar product.
3) Definition of scalar product and Hilbert space.

Consider a string tied at the ends. Use a-dimensional variables. Then:

$$u_{tt} - u_{xx} = 0 \quad \text{and} \quad u = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L$$
Find normal modes (or separate variables), and find connection with characteristics: Normal modes as superpositions of a right and a left traveling wave.

Wave equation. Show that:
Boundary conditions for a tied string of length L lead to a solution of space period \( P = 2L \) --- extend solution "reflecting" across ends.

Example of normal modes.
\[
T_t = T_{xx} \quad \text{for } 0 < x < 1, \quad \text{with:} \\
T = 0 \quad \text{for } x = 0, \quad \text{and} \\
T_x + T = 0 \quad \text{for } x = 1.
\]

Physical meaning of the boundary conditions (heat and elasticity)
- Dirichlet: ice bath or rigid clamped end.
- Neumann: flux prescribed or free end (no stress).
- Robin: fluid cooling or elastically clamped end.

Show space operator in \([E]\) is self-adjoint and negative \( \Rightarrow \) eigenvalues real and negative.
Calculate eigenvalues and use to write solution in terms of the initial value \( T(x, 0) = f(x) \).
Graphical solution of the equation for the eigenvalues: \( \lambda = -k^2 \), where \( k \cos(k) + \sin(k) = 0 \), and \( k > 0 \).
Plot \( k \) versus \( \tan(k) \) and show solutions \( k_n, k_n \sim \pi(n-1/2), n = 1, 2, ... \)
Explain how to solve using Newton's method.

Brief description of separation of variables, and do example \( u_{xx} + u_{yy} = 0 \) for \( r < 1 \) and \( u \) given on \( r = 1 \).
Use polar coordinates, so \( r(ru_r)_r + u_{\theta \theta} \theta = 0 \).
Point out method works only for some equations in some coordinate systems.

Students should read the "short notes on separation of variables" included with problem set #7.

Normal modes. Example of heat equation in \( 0 < x < \pi \), with zero BC.

Linear algebra review: begin with self-adjoint and scalar products.