Course 18.312: Algebraic Combinatorics

In-Class Exam # 2

April 17, 2009

Open notes. Closed Friends and Enemies. No calculators, computers, I-pods, or Zunes. Please explain your reasoning or method, even for computational problems. You may do the problems in any order. There is a total of 100 points. Good Luck.

0) (5 points) Please state a tentative title of your final project or a one-two sentence description.

1) Consider the full binary tree $T$ containing 4 leaves (seven vertices in all). Consider coloring of $T$ up to isomorphism. The symmetry group of $T$ is a group of order 8.

(15 points) a) What is the cycle-index polynomial of $G$ acting on the vertices of $T$?

(10 points) b) In how many ways can the vertices of $T$ be colored in $n$ colors up to reflective symmetry?

2) (20 points) In how many ways can we begin with the empty partition $\emptyset$, then add $2n$ squares one at a time (always keeping a partition), then remove $n$ squares at a time, then add $n$ squares at a time, and finally remove $2n$ squares one at a time, ending up at $\emptyset$?

3) Let $G$ be a regular loopless (undirected) graph of degree $d$ with $p$ vertices and $q$ edges.

(5 points) a) Find a simple relation between $p$, $q$, and $d$.

(5 points) b) Express the biggest eigenvalue of the adjacency matrix $A$ of $G$ in terms of $p$, $q$, and $d$.

(You may use the fact from matrix theory that if $M$ is a $p \times p$ matrix whose entries are nonnegative real numbers, and if $z$ is a column vector of positive
real numbers such that $Mz = \lambda z$, then $\lambda$ is the largest (in absolute value) eigenvalue of $M$.

(5 points) c) Suppose that $G$ has no multiple edges. Express the number of closed walks in $G$ of length two in terms of $p, q,$ and $d$.

Suppose that $G$ has no multiple edges and that the number of closed walks in $G$ of length $\ell$ is given by

$$4^\ell + 5(-2)^\ell + 3 \cdot 2^\ell.$$

(10 points) d) Find the number $\kappa(G)$ of spanning trees of $G$. (Don’t forget that $A$ may have some eigenvalues equal to 0.) For full credit, give a purely numerical answer, not involving $p$, $q$, or $d$, but leaving exponents in your expression is okay.

4) Let $f(n)$ denote the number of permutations in the symmetric group $S_n$, all of whose cycles have length divisible by three.

(15 points) a) Let

$$F(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!}.$$

Find a simple expression for $F(x)$. For full credit, your answer should not involve any summation symbols (or their equivalent), logarithms, or the function $e^x$.

(Bonus 5 points) b) Use part (a) to find a formula for $f(n)$. The answer should be expressed in terms of a binomial coefficient $(r \choose s)$ where $r$ need not be an integer, but $s$ is a nonnegative integer.

5) (5 points) a) Write the elementary symmetric function $e_{41}$ as a sum of $h_\lambda$’s.

(5 points) b) Write the Schur function $s_2 - s_{11}$ as a symmetric polynomial in the variables $\{x_1, x_2, x_3\}$.

(Bonus 10 points) c) For $k \geq 2$, write the Schur function $s_k - s_{k-1,1}$ as a symmetric polynomial in the variables $\{x_1, x_2\}$. 
