1 Partially Ordered Sets II: Dilworth’s Theorem

Definition. An antichain $A$ of a poset $P$ is a subset of elements of $P$ such that for all $x, y \in A$, $x \not\leq y$ and $y \not\leq x$.

We denote the levels of a graded poset $P$ as $P_i$ where $P_i = \{x \in P : \text{rank}(x) = i\}$.

Remark. Observe that each $P_i$ is an antichain, thus we have the inequality

$$\max\{|A| : A \text{ is an antichain of } P\} \geq \{|P_i| : 0 \leq i \leq \text{rank}(P)\}.$$  

We say that the poset $P$ has the Sperner Property if this inequality is actually an equality. A combinatorial condition to check if $P$ has the Sperner property is discussed in Section 4 of Richard Stanley’s notes “Topics in Algebraic Combinatorics”.

Definition. A chain cover of a poset $P$ is a collection of chains whose union is $P$.

(Robert) Dilworth’s Theorem 1950. In any finite poset, the minimum size of a chain cover equals the maximum size of an antichain.

Related Proposition. In any finite poset, the minimum size of an antichain cover equals the maximum size of a chain.

Proof. If $C$ is a chain and $A$ is an antichain cover, then no antichain in $A$ can contain more than one element of $C$, i.e. $A = \{A_1, A_2, \ldots, A_k\}$ and $|A_i \cap C| \leq 1$, so $|A| \geq |C|$ = the maximum size of a chain.

On the other hand, we can define the following antichain cover to show that equality is actually achieved:

$$A_i = \{x \in P \text{ s.t. the longest chain with maximal element } x \text{ has length } i\}.$$
These $A_i$’s are like the levels $P_i$ however we do not need to assume that $P$ is graded. Consequently, $|A|$ is the size of the longest chain, by construction.

**Proof of Dilworth’s Theorem.** Turning to the other situation, notice that if we pick a chain cover $C = \{C_1, C_2, \ldots, C_k\}$ where these chains are pairwise disjoint, then for any antichain $A$, we have $|A \cap C_i| \leq 1$, and thus $|C| \geq |A|$. Thus the maximum size of a chain cover is greater than or equal to the maximum size of an antichain. However, unlike the related proposition, this time the equality is not as easy to see. We follow an inductive proof based on a proof of Fred Galvin (1999).

Firstly, Dilworth’s theorem is trivial if $P$ is empty, so we may assume that $P$ contains at least one element and we therefore let $a$ denote a maximal element of $P$. By induction, we have that Dilworth’s theorem holds for $P' = P \setminus \{a\}$, thus there exist $k$ disjoint chains $C_1, \ldots, C_k$ and at least one antichain $A_0$ of size $k$ such that $A_0 \cap C_i \neq \emptyset$ for $i = 1, 2, \ldots, k$. (If there were to exist a $C_i$ such that $A_0 \cap C_i = \emptyset$, then we would have $|A_0| < |C|$, going against our assumption that $|A_0| = k = |C|$.)

We then use $A_0$’s existence to build another antichain: let $x_i$ be the maximal element in $C_i$ that belongs to an antichain of length $k$ in $P'$. (Note that if $A_0$ was the only antichain of size $k$ then $\{x_1, \ldots, x_k\}$ would be $A_0$, but otherwise $A := \{x_1, \ldots, x_k\}$ could be a different set.)

**Claim.** This set that we just built, $A$, is an antichain.

**Proof.** If $x_j \geq x_i$, then the maximal element of $C_i$ belonging to an antichain of length $k$ would be comparable to the maximal element of $C_j$ belonging to an antichain of length $k$. However, then the set $A'$ which contains $x_j$ and is an antichain of length $k$ would satisfy $A' \cap C_i \neq \emptyset$ so there would exist $y \in A' \cap C_i$. But then, $y$ would be in an antichain of length $k$ and $x_i \geq y$ by $x_i$’s maximality, and we get a contradiction since $x_j$ and $y$ were supposed to both be part of the antichain $A'$.

We finish the proof of Dilworth’s Theorem by reducing to two cases: (1) If $P$ contains a chain cover of size $(k + 1)$ (larger is impossible), then by the maximality of $a$, we must have that $\{a\}$ is a chain. Otherwise, $P \setminus \{a\}$ would also contain a chain cover of size $(k + 1)$ and we would have a contradiction. In this case, $\{a\} \cup A$ is an antichain of size $(k + 1)$. 
(2) On the other hand, if \( \{a\} \cup C_i \) is a chain of \( P \), then \( a \geq x_i \) where \( x_i \in A \) was defined to be the maximal element of chain \( C_i \) (in \( P \backslash \{a\} \)) contained in an antichain of size \( k \). We let \( C' = \{a\} \cup \{y \in C_i : y \leq x_i\} \). The poset \( P \backslash C' \) cannot contain an antichain of size \( k \), only one of size \((k - 1)\). Therefore by induction and Dilworth’s Theorem, \( P \backslash C' \) can be covered by \((k - 1)\) but not \( k \), disjoint chains. Therefore, in this case, \( P \)’s largest antichain is of size \( k \) and largest chain cover is also of size \( k \).
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