1. (a) We have

\[
\sum_{n \geq 0} f(n) x^n = \prod_{k \geq 0} \left( 1 + x^{2^k} + x^{2^k} + x^{3 \cdot 2^k} \right)
\]

\[
= \prod_{k \geq 0} \frac{1 - x^{4 \cdot 2^k}}{1 - x^{2^k}}.
\]

The numerator factors cancel all the denominator factors except the first two, i.e., \(1 - x\) and \(1 - x^2\), so

\[
\sum_{n \geq 0} f(n) x^n = \frac{1}{(1 - x)(1 - x^2)}.
\]

Hence \(f(n)\) is equal to the number of partitions of \(n\) with parts 1 and 2, so \(S = \{1, 2\}\).

(b) We want to count partitions of \(n\) into parts 1 and 2. The number of 2’s in the partition can range from 0 to \(\lfloor n/2 \rfloor\), and the remaining parts must equal 1. Hence the number of choices is \(1 + \lfloor n/2 \rfloor\).

2. Multiply the recurrence by \(x^{n+2}\) and sum on \(n \geq 0\). Set \(F(x) = \sum_{n \geq 0} a_n x^n\). We get

\[
F(x) - x = 6xF(x) - 8F(x),
\]

so

\[
F(x) = \frac{x}{1 - 6x + 8x^2}
\]

\[
= \frac{x}{(1 - 2x)(1 - 4x)}
\]

\[
= \frac{1/2}{1 - 4x} - \frac{1/2}{1 - 2x}.
\]

Hence \(f(n) = \frac{1}{2}(4^n - 2^n)\), Since \(\frac{1}{2}(4^n - 2^n) = \frac{1}{2}2^n(2^n - 1)\), \(f(n)\) is always a triangular number.
3. We are choosing an ordered pair \((S, T)\) of subsets of the pencils such that \(#S\) is odd and then coloring each pencil in \(S\) either red, blue, green, or yellow, and coloring each pencil in \(T\) either white, black, or Halayà úbe. If \(S\) has \(k\) elements where \(k\) is odd, then the number of colorings of \(S\) is \(4^k\). If \(T\) has \(k\) elements then the number of colorings of \(T\) is \(3^k\). The exponential generating function for the number of colorings of \(S\) is

\[
F(x) = \sum_{k \text{ odd}} 4^k \frac{x^k}{k!} = \frac{1}{2} (e^{4x} - e^{-4x}).
\]

The exponential generating function for the number of colorings of \(T\) is

\[
G(x) = \sum_{k \geq 0} 3^k \frac{x^k}{k!} = e^{3x}.
\]

Hence by Theorem 8.21 on page 168, we have

\[
\sum_{n \geq 0} f(n) \frac{x^n}{n!} = \frac{1}{2} (e^{4x} - e^{-4x}) e^{3x}
\]

\[
= \frac{1}{2} (e^{7x} - e^{-x})
\]

\[
= \frac{1}{2} \sum_{n \geq 0} (7^n - (-1)^n) \frac{x^n}{n!},
\]

so \(f(n) = \frac{1}{2} (7^n - (-1)^n)\).


4. (a) Each Hamiltonian path has \(n - 1\) edges. The total number of edges of \(K_n\) is \(\binom{n}{2} = n(n - 1)/2\). Hence the number of paths is \(n/2\), which fails to be an integer when \(n\) is odd.

(b) Let the vertices be 0, 1, \ldots, \(n - 1\). Let one of the paths \(P\) have vertices (in their order along \(P\))

\[0, n - 1, 1, n - 2, 2, n - 3, \ldots, \frac{n}{2}\]
Let the other paths be obtained from $P$ by adding $i$ to each coordinate for $i = 1, 2, \ldots, \frac{n}{2} - 1$, and taking the sum modulo $n$ (i.e., if the sum exceeds $n - 1$ then subtract $n$ from it). For instance, if $n = 8$ then the four paths are

\begin{align*}
0 &\quad 7 &\quad 1 &\quad 6 &\quad 2 &\quad 5 &\quad 3 &\quad 4 \\
1 &\quad 0 &\quad 2 &\quad 7 &\quad 3 &\quad 6 &\quad 4 &\quad 5 \\
2 &\quad 1 &\quad 3 &\quad 0 &\quad 4 &\quad 7 &\quad 5 &\quad 6 \\
3 &\quad 2 &\quad 4 &\quad 1 &\quad 5 &\quad 0 &\quad 6 &\quad 7.
\end{align*}

We leave the verification that this works as an exercise.

Another way to describe the same solution (suggested by Y. Hu) is to put the vertices $0, 1, \ldots, n - 1$ in clockwise order on a circle. Let $P$ be the zigzag path whose vertices (in order) are $0, n - 1, 1, n - 2, 2, n - 3, 3, \ldots, \frac{n}{2}$. Rotate the circle around the center by $2j\pi/n$ radians for $0 \leq j \leq \frac{n}{2} - 1$. Each such rotation gives one of the paths in the partition into Hamiltonian paths.