Exercises 6

(1) Let $A$ be a central arrangement in $\mathbb{R}^n$ with distance enumerator $D_A(t)$ (with respect to some base region $R_0$). Define a graph $G_A$ on the vertex set $\mathcal{R}(A)$ by putting an edge between $R$ and $R'$ if $\#\text{sep}(R, R') = 1$ (i.e., $R$ and $R'$ are separated by a unique hyperplane).

(a) [2–] Show that $G_A$ is a bipartite graph.

(b) [2] Show that if $\#A$ is odd, then $D_A(-1) = 0$.

(c) [2] Show that if $\#A$ is even and $r(A) \equiv 2 \pmod{4}$, then $D_A(-1) \equiv 2 \pmod{4}$ (so $D_A(-1) \neq 0$).

(d) [2] Give an example of (c), i.e., find $A$ so that $\#A$ is even and $r(A) \equiv 2 \pmod{4}$.

(e) [2] Show that (c) cannot hold if $A$ is supersolvable. (It is not assumed that the base region $R_0$ is canonical. Try to avoid the use of Section 1.6.4.)

(f) [2+] Show that if $\#A$ is even and $r(A) \equiv 0 \pmod{4}$, then it is possible for $D_A(-1) = 0$ and for $D_A(-1) \neq 0$. Can examples be found for $\text{rank}(A) \leq 3$?

(2) [2–] Show that a sequence $(c_1, \ldots, c_n) \in \mathbb{N}^n$ is the inversion sequence of a permutation $w \in \Theta_n$ if and only if $c_i \leq i - 1$ for $1 \leq i \leq n$.

(3) [2] Show that all cars can park under the scenario following Definition 1.1 if and only if the sequence $(a_1, \ldots, a_n)$ of preferred parking spaces is a parking function.

(4) [5] Find a bijective proof of Theorem 1.2, i.e., find a bijection $\varphi$ between the set of all rooted forests on $[n]$ and the set $\text{PF}_n$ of all parking functions of length $n$ satisfying $\text{inv}(F) = \binom{n+1}{2} - a_1 - \cdots - a_n$ when $\varphi(F) = (a_1, \ldots, a_n)$. Note. In principle a bijection $\varphi$ can be obtained by carefully analyzing the proof of Theorem 1.2. However, this bijection will be of a messy recursive nature. A "nonrecursive" bijection would be greatly preferred.

(5) [5] There is a natural two-variable refinement of the distance enumerator (9) of $\text{Sn}$. Given $R \in \mathcal{R}(S_n)$, define $d_0(R_0, R)$ to be the number of hyperplanes $x_i = x_j$ separating $R_0$ from $R$, and $d_1(R_0, R)$ to be the number of hyperplanes $x_i = x_j + 1$ separating $R_0$ from $R$. (Here $R_0$ is given by (7) as usual.) Set

$$D_n(q, t) = \sum_{R \in \mathcal{R}(S_n)} q^{d_0(R_0, R)} t^{d_1(R_0, R)}.$$

What can be said about the polynomial $D_n(q, t)$? Can its coefficients be interpreted in a simple way in terms of tree or forest inversions? Are there formulas or recurrences for $D_n(q, t)$ generalizing Theorem 1.1, Corollary 1.1, or equation (5)? The table below give the coefficients of $q^i t^j$ in $D_n(q, t)$ for $2 \leq n \leq 4$.

\[
\begin{array}{c|ccc}
  q \backslash t^1 & 0 & 1 & 2 \\
  \hline
  0 & 1 & 1 & 1 \\
  1 & 1 & 1 & 1 \\
  2 & 2 & 1 & 1 \\
  3 & 1 & 1 & 1 \\
  4 & 1 & 1 & 1 \\
  5 & 1 & 1 & 1 \\
  6 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|ccccc}
  q \backslash t^1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
  \hline
  0 & 1 & 1 & 2 & 3 & 3 & 3 & 3 \\
  1 & 1 & 3 & 6 & 7 & 6 & 3 & 3 \\
  2 & 5 & 8 & 9 & 5 & 6 & 7 & 6 \\
  3 & 6 & 7 & 9 & 6 & 5 & 6 & 5 \\
  4 & 5 & 6 & 5 & 6 & 5 & 6 & 5 \\
  5 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
  6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
Some entries of these table are easy to understand, e.g., the first and last entries in each row and column, but a simple way to compute the entire table is not known.

(6) [5–] Let $G_n$ denote the generic braid arrangement

\[ x_i - x_j = a_{ij}, \quad 1 \leq i < j \leq n, \]

in $\mathbb{R}^n$. Can anything interesting be said about the distance enumerator $D_{G_n}(t)$ (which depends on the choice of base region $R_0$ and possibly on the $a_{ij}$’s)? Generalize if possible to generic graphical arrangements, especially for supersolvable (or chordal) graphs.

(7) [3–] Let $A$ be a real supersolvable arrangement and $R_0$ a canonical region of $A$. Show that the weak order $W_A$ (with respect to $R_0$) is a lattice.

(8) (a) [2+] let $A$ be a real central arrangement of rank $d$. Suppose that the weak order $W_A$ (with respect to some region $R_0 \in \mathcal{R}(A)$) is a lattice. Show that $R_0$ is simplicial, i.e., bounded by exactly $d$ hyperplanes.

(b) [3–] Let $A$ be a real central arrangement. Show that if every region $R \in \mathcal{R}(A)$ is simplicial, then $W_A$ is a lattice.