Lecture 4
Broken circuits, modular elements, and supersolvability

This lecture is concerned primarily with matroids and geometric lattices. Since the intersection lattice of a central arrangement is a geometric lattice, all our results can be applied to arrangements.

4.1. Broken circuits

For any geometric lattice $L$ and $x \leq y$ in $L$, we have seen (Theorem 3.10) that $(-1)^{rk(x,y)}\mu(x,y)$ is a positive integer. It is thus natural to ask whether this integer has a direct combinatorial interpretation. To this end, let $M$ be a matroid on the set $S = \{u_1, \ldots, u_m\}$. Linearly order the elements of $S$, say $u_1 < u_2 < \cdots < u_m$.

Recall that a circuit of $M$ is a minimal dependent subset of $S$.

Definition 4.10. A broken circuit of $M$ (with respect to the linear ordering $O$ of $S$) is a set $C - \{u\}$, where $C$ is a circuit and $u$ is the largest element of $C$ (in the ordering $O$). The broken circuit complex $BC_O(M)$ (or just $BC(M)$ if no confusion will arise) is defined by

$$BC(M) = \{T \subseteq S : T \text{ contains no broken circuit}\}.$$  

Figure 1 shows two linear orderings $O$ and $O'$ of the points of the affine matroid $M$ of Figure 1 (where the ordering of the points is $1 < 2 < 3 < 4 < 5$). With respect to the first ordering $O$ the circuits are $123, 345, 1245$, and the broken circuits are $12, 34, 124$. With respect to the second ordering $O'$ the circuits are $123, 145, 2345$, and the broken circuits are $12, 14, 234$.

It is clear that the broken circuit complex $BC(M)$ is an abstract simplicial complex, i.e., if $T \in BC(M)$ and $U \subseteq T$, then $U \in BC(M)$. In Figure 1 we
have $\text{BC}_\varnothing(M) = \langle 135, 145, 235, 245 \rangle$, while $\text{BC}_{\varnothing'}(M) = \langle 135, 235, 245, 345 \rangle$. These simplicial complexes have geometric realizations as follows:

Note that the two simplicial complexes $\text{BC}_\varnothing(M)$ and $\text{BC}_{\varnothing'}(M)$ are not isomorphic (as abstract simplicial complexes); in fact, their geometric realizations are not even homeomorphic. On the other hand, if $f_i(\Delta)$ denotes the number of $i$-dimensional faces (or faces of cardinality $i - 1$) of the abstract simplicial complex $\Delta$, then for $\Delta$ given by either $\text{BC}_\varnothing(M)$ or $\text{BC}_{\varnothing'}(M)$ we have

$$f_{-1}(\Delta) = 1, \ f_0(\Delta) = 5, \ f_1(\Delta) = 8, \ f_2(\Delta) = 4.$$ 

Note, moreover, that

$$\chi_M(t) = t^4 - 5t^2 + 8t - 4.$$ 

In order to generalize this observation to arbitrary matroids, we need to introduce a fair amount of machinery, much of it of interest for its own sake. First we give a fundamental formula, known as Philip Hall’s theorem, for the Möbius function value $\mu(\hat{0}, 1)$.

**Lemma 4.4.** Let $P$ be a finite poset with $\hat{0}$ and $\hat{1}$, and with Möbius function $\mu$. Let $c_i$ denote the number of chains $0 = y_0 < y_1 < \cdots < y_i = \hat{1}$ in $P$. Then

$$\mu(\hat{0}, \hat{1}) = -c_1 + c_2 - c_3 + \cdots.$$

**Proof.** We work in the incidence algebra $\mathfrak{I}(P)$. We have

$$\mu(\hat{0}, \hat{1}) = \zeta^{-1}(\hat{0}, \hat{1})$$

$$= (\delta + (\zeta - \delta))^{-1}(\hat{0}, \hat{1})$$

$$= \delta(\hat{0}, \hat{1}) - (\zeta - \delta)(\hat{0}, \hat{1}) + (\zeta - \delta)^2(\hat{0}, \hat{1}) - \cdots.$$ 

This expansion is easily justified since $(\zeta - \delta)^k(\hat{0}, \hat{1}) = 0$ if the longest chain of $P$ has length less than $k$. By definition of the product in $\mathfrak{I}(P)$ we have $(\zeta - \delta)^c(\hat{0}, \hat{1}) = c_i$, and the proof follows. □

**Note.** Let $P$ be a finite poset with $\hat{0}$ and $\hat{1}$, and let $P' = P - \{\hat{0}, \hat{1}\}$. Define $\Delta(P')$ to be the set of chains of $P'$, so $\Delta(P')$ is an abstract simplicial complex. The reduced Euler characteristic of a simplicial complex $\Delta$ is defined by

$$\check{\chi}(P) = -f_{-1} + f_0 - f_1 + \cdots,$$

where $f_i$ is the number of $i$-dimensional faces $F \in \Delta$ (or $\#F = i + 1$). Comparing with Lemma 4.4 shows that

$$\mu(\hat{0}, \hat{1}) = \check{\chi}(\Delta(P')).$$

Readers familiar with topology will know that $\check{\chi}(\Delta)$ has important topological significance related to the homology of $\Delta$. It is thus natural to ask whether results
concerning Möbius functions can be generalized or refined topologically. Such results are part of the subject of “topological combinatorics,” about which we will say a little more later.

Now let $P$ be a finite graded poset with $\hat{0}$ and $\hat{1}$. Let

$$\mathcal{E}(P) = \{(x, y) : x \leq y \text{ in } P\},$$

the set of (directed) edges of the Hasse diagram of $P$.

**Definition 4.11.** An $E$-labeling of $P$ is a map $\lambda : \mathcal{E}(P) \to \mathbb{P}$ such that if $x \leq y$ in $P$ then there exists a unique saturated chain

$$C : x = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_k = y$$

satisfying

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k).$$

We call $C$ the increasing chain from $x$ to $y$.

Figure 2 shows three examples of posets $P$ with a labeling of their edges, i.e., a map $\lambda : \mathcal{E}(P) \to \mathbb{P}$. Figure 2(a) is the boolean algebra $B_3$ with the labeling $\lambda(S, S \cup \{i\}) = i$. (The one-element subsets $\{i\}$ are also labelled with a small $i$.) For any boolean algebra $B_n$, this labeling is the archetypal example of an $E$-labeling. The unique increasing chain from $S$ to $T$ is obtained by adjoining to $S$ the elements of $T - S$ one at a time in increasing order. Figures 2(b) and (c) show two different $E$-labelings of the same poset $P$. These labelings have a number of different properties, e.g., the first has a chain whose edge labels are not all different, while every maximal chain label of Figure 2(c) is a permutation of $\{1, 2\}$.

**Theorem 4.11.** Let $\lambda$ be an $E$-labeling of $P$, and let $x \leq y$ in $P$. Let $\mu$ denote the Möbius function of $P$. Then $(-1)^{rk(x, y)}\mu(x, y)$ is equal to the number of strictly decreasing saturated chains from $x$ to $y$, i.e.,

$$(-1)^{rk(x, y)}\mu(x, y) =$$

$$\#\{x = x_0 \prec x_1 \prec \cdots \prec x_k = y : \lambda(x_0, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_{k-1}, x_k)\}.$$ 

**Proof.** Since $\lambda$ restricted to $[x, y]$ (i.e., to $\mathcal{E}([x, y])$) is an $E$-labeling, we can assume $[x, y] = [\hat{0}, \hat{1}] = P$. Let $S = \{a_1, a_2, \ldots, a_{j-1}\} \subseteq [n - 1]$, with $a_1 < a_2 < \cdots < a_{j-1}$. 

![Figure 2. Three examples of edge-labelings](image)
Define \( \alpha_P(S) \) to be the number of chains \( \hat{0} < y_1 < \cdots < y_{j-1} < \hat{1} \) in \( P \) such that \( \text{rk}(y_i) = a_i \) for \( 1 \leq i \leq j-1 \). The function \( \alpha_P \) is called the flag f-vector of \( P \).

Claim. \( \alpha_P(S) \) is the number of maximal chains \( \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1} \) such that

\[
\lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1}) \Rightarrow i \in S, \ 1 \leq i \leq n.
\]

To prove the claim, let \( \hat{0} = y_0 < y_1 < \cdots < y_{j-1} < y_j = \hat{1} \) with \( \text{rk}(y_i) = a_i \) for \( 1 \leq i \leq j-1 \). By the definition of \( E \)-labeling, there exists a unique refinement

\[
\hat{0} = y_0 = x_0 < x_1 < \cdots < x_{a_1} = y_1 < x_{a_1+1} < \cdots < x_{a_2} = y_2 < \cdots < x_n = y_j = \hat{1}
\]

satisfying

\[
\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{a_1-1}, x_{a_1})
\]

\[
\lambda(x_{a_1}, x_{a_1+1}) \leq \lambda(x_{a_1+1}, x_{a_1+2}) \leq \cdots \leq \lambda(x_{a_2-1}, x_{a_2})
\]

\[
\cdots
\]

Thus if \( \lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1}) \), then \( i \in S \), so (27) is satisfied. Conversely, given a maximal chain \( \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1} \) satisfying the above conditions on \( \lambda \), let \( y_i = x_{a_i} \). Therefore we have a bijection between the chains counted by \( \alpha_P(S) \) and the maximal chains satisfying (27), so the claim follows.

Now for \( S \subseteq [n-1] \) define

\[
\beta_P(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_P(T).
\]

The function \( \beta_P \) is called the flag h-vector of \( P \). A simple Inclusion-Exclusion argument gives

\[
\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T),
\]

for all \( S \subseteq [n-1] \). It follows from the claim and equation (29) that \( \beta_P(T) \) is equal to the number of maximal chains \( \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1} \) such that \( \lambda(x_i) > \lambda(x_{i+1}) \) if and only if \( i \in T \). In particular, \( \beta_P([n-1]) \) is equal to the number of strictly decreasing maximal chains \( \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1} \) of \( P \), i.e.,

\[
\lambda(x_0, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_{n-1}, x_n).
\]

Now by (28) we have

\[
\beta_P([n-1]) = \sum_{T \subseteq [n-1]} (-1)^{n-1-\#T} \alpha_P(T)
\]

\[
= \sum_{k \geq 1} \sum_{0=y_0 < y_1 < \cdots < y_k=\hat{1}} (-1)^{n-k}
\]

\[
= (-1)^n \sum_{k \geq 1} (-1)^k c_k,
\]

where \( c_k \) is the number of chains \( \hat{0} = y_0 < y_1 < \cdots < y_i = \hat{1} \) in \( P \). The proof now follows from Philip Hall’s theorem (Lemma 4.4).

We come to the main result of this subsection, a combinatorial interpretation of the coefficients of the characteristic polynomial \( \chi_M(t) \) for any matroid \( M \).
Theorem 4.12. Let $M$ be a matroid of rank $n$ with a linear ordering $x_1 < x_2 < \cdots < x_m$ of its points (so the broken circuit complex $BC(M)$ is defined), and let $0 \leq i \leq n$. Then

$$(-1)^i\ell^n|\chi_M(t) = f_{i-1}(BC(M)).$$

**Proof.** We may assume $M$ is simple since the “simplification” $\hat{M}$ has the same lattice of flats and same broken circuit complex as $M$ (Exercise 1). The atoms $x_i$ of $L(M)$ can then be identified with the points of $M$. Define a labeling $\lambda : E(L(M)) \rightarrow \mathbb{P}$ as follows. Let $x < y$ in $L(M)$. Then set

$$(30) \quad \lambda(x, y) = \max\{i : x \lor x_i = y\}.$$ 

Note that $\lambda(x, y)$ is defined since $L(M)$ is atomic.

As an example, Figure 3 shows the lattice of flats of the matroid $M$ of Figure 1 with the edge labeling $(30)$.

**Claim 1.** Define $\lambda : E(L(M)) \rightarrow \mathbb{P}$ by

$$\lambda(x, y) = m + 1 - \lambda(x, y).$$

Then $\lambda$ is an $E$-labeling.

To prove this claim, we need to show that for all $x < y$ in $L(M)$ there is a unique saturated chain $x = y_0 < y_1 < \cdots < y_k = y$ satisfying

$$\lambda(y_0, y_1) \geq \lambda(y_1, y_2) \geq \cdots \geq \lambda(y_{k-1}, y_k).$$

The proof is by induction on $k$. There is nothing to prove for $k = 1$. Let $k > 1$ and assume the assertion for $k - 1$. Let

$$j = \max\{i : x_i \leq y, x_i \not\in x\}.$$ 

For any saturated chain $x = z_0 < z_1 < \cdots < z_k = y$, there is some $i$ for which $x_j \not\in z_i$ and $x_j \leq z_{i+1}$. Hence $\lambda(z_i, z_{i+1}) = j$. Thus if $\lambda(z_0, z_1) \geq \cdots \geq \lambda(z_{k-1}, z_k)$, then $\lambda(z_0, z_1) = j$. Moreover, there is a unique $y_1$ satisfying $x = x_0 < y_1 \leq y$ and $\lambda(x_0, y_1) = j$, viz., $y_1 = x_0 \lor x_j$. (Note that $y_1 > x_0$ by semimodularity.)
By the induction hypothesis there exists a unique saturated chain 
\( y_1 < y_2 < \cdots < y_k = y \) satisfying \( \lambda(y_1,y_2) \geq \cdots \geq \lambda(y_{k-1},y_k) \). Since \( \lambda(y_0,y_1) = j > \lambda(y_1,y_2) \), the proof of Claim 1 follows by induction.

**Claim 2.** The broken circuit complex \( BC(M) \) consists of all chain labels \( \lambda(C) \), where \( C \) is a saturated increasing chain (with respect to \( \lambda \)) from \( 0 \) to some \( x \in L(M) \). Moreover, all such \( \lambda(C) \) are distinct.

To prove the distinctness of the labels \( \lambda(C) \), suppose that \( C \) is given by \( \hat{0} = y_0 < y_1 < \cdots < y_k \), with \( \lambda(C) = (a_1,a_2,\ldots,a_k) \). Then \( y_i = y_{i-1} \lor x_{a_i} \), so \( C \) is the only chain with its label.

Now let \( C \) and \( \lambda(C) \) be as in the previous paragraph. We claim that the
set \( \{x_{a_1},\ldots,x_{a_k}\} \) contains no broken circuit. (We don’t even require that \( C \) is increasing for this part of the proof.) Write \( z_i = x_{a_i} \), and suppose to the contrary that \( B = \{z_{i_1},\ldots,z_{i_j}\} \) is a broken circuit, with \( 1 \leq i_1 < \cdots < i_j \leq k \). Let \( B \cup \{x_r\} \) be a circuit with \( r > a_{i_t} \) for \( 1 \leq t \leq j \). Now for any circuit \( \{u_1,\ldots,u_h\} \) and any \( 1 \leq i \leq h \) we have

\[
u_1 \lor u_2 \lor \cdots \lor u_h = u_1 \lor \cdots \lor u_{i-1} \lor u_{i+1} \lor \cdots \lor u_h.
\]

Thus

\[
z_{i_1} \lor z_{i_2} \lor \cdots \lor z_{i_{j-1}} \lor x_r = \bigvee_{z \in B} z = z_{i_1} \lor z_{i_2} \lor \cdots \lor z_{i_j}.
\]

Then \( y_{i_{j-1}} \lor x_r = y_{i_j} \), contradicting the maximality of the label \( a_{i_j} \). Hence \( \{x_{a_1},\ldots,x_{a_k}\} \in BC(M) \).

Conversely, suppose that \( T := \{x_{a_1},\ldots,x_{a_k}\} \) contains no broken circuit, with
\( a_1 < \cdots < a_k \). Let \( y_i = x_{a_i} \lor x_{a_i} \lor \cdots \lor x_{a_i} \), and let \( C \) be the chain \( \hat{0} := y_0 < y_1 < \cdots < y_k \). (Note that \( C \) is saturated by semimodularity.) We claim that \( \lambda(C) = (a_1,\ldots,a_k) \).

If not, then \( y_i \lor x_j = y_i \) for some \( j > a_i \). Thus

\[
\text{rk}(T) = \text{rk}(T \cup \{x_j\}) = i.
\]

Since \( T \) is independent, \( T \cup \{x_j\} \) contains a circuit \( Q \) satisfying \( x_j \in Q \), so \( T \) contains a broken circuit. This contradiction completes the proof of Claim 2.

To complete the proof of the theorem, note that we have shown that \( f_{r-1}(BC(M)) \) is the number of chains \( C : \hat{0} = y_0 < y_1 < \cdots < y_i \) such that \( \lambda(C) \) is strictly increasing, or equivalently, \( \lambda(C) \) is strictly decreasing. Since \( \lambda \) is an \( E \)-labeling, the proof follows from Theorem 4.11. \( \square \)

**Corollary 4.6.** The broken circuit complex \( BC(M) \) is pure, i.e., every maximal face has the same dimension.

to be inserted. \( \square \)

Note (for readers with some knowledge of topology). (a) Let \( M \) be a matroid on the linearly ordered set \( u_1 < u_2 < \cdots < u_m \). Note that \( F \in BC(M) \) if and only if \( F \cup \{u_m\} \in BC(M) \). Define the reduced broken circuit complex \( BC_r(M) \) by

\[
BC_r(M) = \{F \in BC(M) : u_m \notin F\}.
\]

Thus

\[
BC(M) = BC_r(M) \ast u_m,
\]

the join of \( BC_r(M) \) and the vertex \( u_m \). Equivalently, \( BC(M) \) is a cone over \( BC_r(M) \) with apex \( u_m \). As a consequence, \( BC(M) \) is contractible and therefore has the homotopy type of a point. A more interesting problem is to determine the topological nature of \( BC_r(M) \). It can be shown that \( BC_r(M) \) has the homotopy type of a wedge
of \( \beta(M) \) spheres of dimension \( \text{rank}(M) - 2 \), where \((-1)^{\text{rank}(M)-1}\beta(M) = \chi'_M(1) \) (the derivative of \( \chi_M(t) \) at \( t = 1 \)). See Exercise 21 for more information on \( \beta(M) \). 

(b) [to be inserted]

As an example of the applicability of our results on matroids and geometric lattices to arrangements, we have the following purely combinatorial description of the number of regions of a real central arrangement.

**Corollary 4.7.** Let \( \mathcal{A} \) be a central arrangement in \( \mathbb{R}^n \), and let \( M \) be the matroid defined by the normals to \( H \in \mathcal{A} \), i.e., the independent sets of \( M \) are the linearly independent normals. Then with respect to any linear ordering of the points of \( M \), \( r(\mathcal{A}) \) is the total number of subsets of \( M \) that don’t contain a broken circuit.

**Proof.** Immediate from Theorems 2.5 and 4.12. \( \Box \)

### 4.2. Modular elements

We next discuss a situation in which the characteristic polynomial \( \chi_M(t) \) factors in a nice way.

**Definition 4.12.** An element \( x \) of a geometric lattice \( L \) is modular if for all \( y \in L \) we have

\[
\text{rk}(x) + \text{rk}(y) = \text{rk}(x \wedge y) + \text{rk}(x \vee y).
\]

**Example 4.9.** Let \( L \) be a geometric lattice.

(a) \( \hat{0} \) and \( \hat{1} \) are clearly modular (in any finite lattice).

(b) We claim that atoms \( a \) are modular.

**Proof.** Suppose that \( a \leq y \). Then \( a \wedge y = a \) and \( a \vee y = y \), so equation (31) holds. (We don’t need that \( a \) is an atom for this case.) Now suppose \( a \not\leq y \). By semimodularity, \( \text{rk}(a \vee y) = 1 + \text{rk}(y) \), while \( \text{rk}(a) = 1 \) and \( \text{rk}(a \wedge y) = \text{rk}(\hat{0}) = 0 \), so again (31) holds.

(c) Suppose that \( \text{rk}(L) = 3 \). All elements of rank 0, 1, or 3 are modular by (a) and (b). Suppose that \( \text{rk}(x) = 2 \). Then \( x \) is modular if and only if for all elements \( y \neq x \) and \( \text{rk}(y) = 2 \), we have that \( \text{rk}(x \wedge y) = 1 \).

(d) Let \( L = B_n \). If \( x \in B_n \) then \( \text{rk}(x) = \# x \). Moreover, for any \( x, y \in B_n \) we have \( x \wedge y = x \cap y \) and \( x \vee y = x \cup y \). Since for any finite sets \( x \) and \( y \) we have

\[
\# x + \# y = \# (x \cap y) + \# (x \cup y),
\]

it follows that every element of \( B_n \) is modular. In other words, \( B_n \) is a modular lattice.

(e) Let \( q \) be a prime power and \( \mathbb{F}_q \) the finite field with \( q \) elements. Define \( B_n(q) \) to be the lattice of subspaces, ordered by inclusion, of the vector space \( \mathbb{F}_q^n \). Note that \( B_n(q) \) is also isomorphic to the intersection lattice of the arrangement of all linear hyperplanes in the vector space \( \mathbb{F}_n(q) \). Figure 4 shows the Hasse diagrams of \( B_2(3) \) and \( B_3(2) \).

Note that for \( x, y \in B_n(q) \) we have \( x \wedge y = x \cap y \) and \( x \vee y = x + y \) (subspace sum). Clearly \( B_n(q) \) is atomic: every vector space is the join (sum) of its one-dimensional subspaces. Moreover, \( B_n(q) \) is graded of rank \( n \), with rank function given by \( \text{rk}(x) = \dim(x) \). Since for any subspaces \( x \) and \( y \) we have

\[
\dim(x) + \dim(y) = \dim(x \cap y) + \dim(x + y),
\]
it follows that $L$ is a modular geometric lattice. Thus every $x \in L$ is modular.

**Note.** A projective plane $R$ consists of a set (also denoted $R$) of points, and a collection of subsets of $R$, called lines, such that: (a) every two points lie on a unique line, (b) every two lines intersect in exactly one point, and (c) (non-degeneracy) there exist four points, no three of which are on a line. The incidence lattice $L(R)$ of $R$ is the set of all points and lines of $R$, ordered by $p < L$ if $p \in L$, with 0 and 1 adjoined. It is an immediate consequence of the axioms that when $R$ is finite, $L(R)$ is a modular geometric lattice of rank 3. It is an open (and probably intractable) problem to classify all finite projective planes. Now let $P$ and $Q$ be posets and define their direct product (or cartesian product) to be the set

$$P \times Q = \{(x,y) : x \in P, y \in Q\},$$

ordered componentwise, i.e., $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. It is easy to see that if $P$ and $Q$ are geometric (respectively, atomic, semimodular, modular) lattices, then so is $P \times Q$ (Exercise 7). It is a consequence of the “fundamental theorem of projective geometry” that every finite modular geometric lattice is a direct product of boolean algebras $B_n$, subspace lattices $B_n(q)$ for $n \geq 3$, lattices of rank 2 with at least five elements (which may be regarded as $B_2(q)$ for any $q \geq 2$) and incidence lattices of finite projective planes.

(f) The following result characterizes the modular elements of $\Pi_n$, which is the lattice of partitions of $[n]$ or the intersection lattice of the braid arrangement $B_n$.

**Proposition 4.9.** A partition $\pi \in \Pi_n$ is a modular element of $\Pi_n$ if and only if $\pi$ has at most one nonsingleton block. Hence the number of modular elements of $\Pi_n$ is $2^n - n$.

**Proof.** If all blocks of $\pi$ are singletons, then $\pi = \hat{0}$, which is modular by (a). Assume that $\pi$ has the block $A$ with $r > 1$ elements, and all other blocks are singletons. Hence the number $|\pi|$ of blocks of $\pi$ is given by

![Diagram of B_2(3) and B_3(2)](image)

*Figure 4. The lattices $B_2(3)$ and $B_3(2)$*
n - r + 1. For any \( \sigma \in \Pi_n \), we have \( \text{rk}(\sigma) = n - |\sigma| \). Let \( k = |\sigma| \) and
\[
j = \# \{ B \in \sigma : A \cap B \neq \emptyset \}.
\]
Then \( |\pi \land \sigma| = j + (n - r) \) and \( |\pi \lor \sigma| = k - j + 1 \). Hence \( \text{rk}(\pi) = r - 1 \), 
\( \text{rk}(\sigma) = n - k \), \( \text{rk}(\pi \land \sigma) = r - j \), and \( \text{rk}(\pi \lor \sigma) = n - k + j - 1 \), so \( \pi \) is modular.

Conversely, let \( \pi = \{ B_1, B_2, \ldots, B_k \} \) with \#\( B_1 \geq 1 \) and \#\( B_2 \geq 1 \).
Let \( a \in B_1 \) and \( b \in B_2 \), and set 
\[
\sigma = \{(B_1 \cup b) - a, (B_2 \cup a) - b, B_3, \ldots, B_k\}.
\]
Then
\[
|\pi| = |\sigma| = k
\]
\[
\pi \land \sigma = \{a, b, B_1 - a, B_2 - b, \ldots, B_3, \ldots, B_k\} \Rightarrow |\pi \land \sigma| = k + 2
\]
\[
\pi \lor \sigma = \{B_1 \cup B_2, B_3, \ldots, B_1\} \Rightarrow |\pi \lor \sigma| = k - 1.
\]
Hence \( \text{rk}(\pi) + \text{rk}(\sigma) \neq \text{rk}(\pi \land \sigma) + \text{rk}(\pi \lor \sigma) \), so \( \pi \) is not modular. \( \square \)

In a finite lattice \( L \), a complement of \( x \in L \) is an element \( y \in L \) such that \( x \land y = 0 \) and \( x \lor y = 1 \). For instance, in the boolean algebra \( B_n \), every element has a unique complement. (See Exercise 3 for the converse.) The following proposition collects some useful properties of modular elements. The proof is left as an exercise (Exercises 4–5).

**Proposition 4.10.** Let \( L \) be a geometric lattice of rank \( n \).

(a) Let \( x \in L \). The following four conditions are equivalent.

(i) \( x \) is a modular element of \( L \).
(ii) If \( x \land y = 0 \), then \( \text{rk}(x) + \text{rk}(y) = \text{rk}(x \lor y) \).
(iii) If \( x \) and \( y \) are complements, then \( \text{rk}(x) + \text{rk}(y) = n \).
(iv) All complements of \( x \) are incomparable.

(b) (Transitivity of modularity) If \( x \) is a modular element of \( L \) and \( y \) is modular in the interval \([0, x]\), then \( y \) is a modular element of \( L \).

(c) If \( x \) and \( y \) are modular elements of \( L \), then \( x \land y \) is also modular.

The next result, known as the modular element factorization theorem [16], is our primary reason for defining modular elements — such an element induces a factorization of the characteristic polynomial.

**Theorem 4.13.** Let \( z \) be a modular element of the geometric lattice \( L \) of rank \( n \).
Write \( \chi_z(t) = \chi_{[0, z]}(t) \). Then
\[
\chi_L(t) = \chi_z(t) \left[ \sum_{y : y \land z = 0} \mu_L(y) t^{n - \text{rk}(y) - \text{rk}(z)} \right].
\]

**Example 4.10.** Before proceeding to the proof of Theorem 4.13, let us consider an example. The illustration below is the affine diagram of a matroid \( M \) of rank 3, together with its lattice of flats. The two lines (flats of rank 2) labelled \( x \) and \( y \) are modular by Example 4.9(c).
Hence by equation (32) $\chi_M(t)$ is divisible by $\chi_x(t)$. Moreover, any atom $a$ of the interval $[\hat{0}, x]$ is modular, so $\chi_x(t)$ is divisible by $\chi_a(t) = t - 1$. From this it is immediate (e.g., because the characteristic polynomial $\chi_G(t)$ of any geometric lattice $G$ of rank $n$ begins $x^n - ax^{n-1} + \cdots$, where $a$ is the number of atoms of $G$) that $\chi_x(t) = (t-1)(t-5)$ and $\chi_M(t) = (t-1)(t-3)(t-5)$. On the other hand, since $y$ is modular, $\chi_M(t)$ is divisible by $\chi_y(t)$, and we get as before $\chi_y(t) = (t-1)(t-3)$ and $\chi_M(t) = (t-1)(t-3)(t-5)$. Geometric lattices whose characteristic polynomial factors into linear factors in a similar way due to a maximal chain of modular elements are discussed further beginning with Definition 4.13.

Our proof of Theorem 4.13 will depend on the following lemma of Greene [11]. We give a somewhat simpler proof than Greene.

**Lemma 4.5.** Let $L$ be a finite lattice with Möbius function $\mu$, and let $z \in L$. The following identity is valid in the Möbius algebra $A(L)$ of $L$:

\[
\sigma_0 := \sum_{x \in L} \mu(x)x = \left( \sum_{v \leq z} \mu(v)v \right) \left( \sum_{y \wedge z = \hat{0}} \mu(y)y \right).
\]
Proof. Let $\sigma_s$ for $s \in L$ be given by (8). The right-hand side of equation (33) is then given by

$$\sum_{v \leq z \atop y \wedge z = 0} \mu(v)\mu(y)(v \vee y) = \sum_{v \leq z \atop y \wedge z = 0} \mu(v)\mu(y) \sum_{s \in v \vee y} \sigma_s$$

$$= \sum_s \sigma_s \sum_{v \leq s, v \leq z \atop y \leq s, y \wedge z = 0} \mu(v)\mu(y)$$

$$= \sum_s \sigma_s \left( \sum_{v \leq s \wedge z} \mu(v) \right) \left( \sum_{y \leq s \wedge z = 0} \mu(y) \right)$$

$$= \sum_{s \wedge z = 0} \sigma_s \left( \sum_{y \leq s \wedge z = 0} \mu(y) \right)$$

$$= \sigma_0.$$

Proof of Theorem 4.13. We are assuming that $z$ is a modular element of the geometric lattice $L$.

Claim 1. Let $v \leq z$ and $y \wedge z = 0$ (so $v \wedge y = 0$). Then $z \wedge (v \vee y) = v$ (as illustrated below).
Proof of Claim 1. Clearly $z \land (v \lor y) \geq v$, so it suffices to show that $\text{rk}(z \land (v \lor y)) \leq \text{rk}(v)$. Since $z$ is modular we have
\[
\text{rk}(z \land (v \lor y)) = \text{rk}(z) + \text{rk}(v \lor y) - \text{rk}(z \lor y) = \text{rk}(z) + \text{rk}(v) - (\text{rk}(z) + \text{rk}(y)) - \text{rk}(z \land y) \\
\leq (\text{rk}(v) + \text{rk}(y) - \text{rk}(v \land y)) - \text{rk}(y) \text{ by semimodularity}
\]
proving Claim 1.

Claim 2. With $v$ and $y$ as above, we have $\text{rk}(v \lor y) = \text{rk}(v) + \text{rk}(y)$.

Proof of Claim 2. By the modularity of $z$ we have
\[
\text{rk}(z \land (v \lor y)) + \text{rk}(z \lor (v \lor y)) = \text{rk}(z) + \text{rk}(v \lor y).
\]
By Claim 1 we have $\text{rk}(z \land (v \lor y)) = \text{rk}(v)$. Moreover, again by the modularity of $z$ we have
\[
\text{rk}(z \lor (v \lor y)) = \text{rk}(z \lor y) = \text{rk}(z) + \text{rk}(y) - \text{rk}(z \land y) = \text{rk}(z) + \text{rk}(y).
\]
It follows that $\text{rk}(v) + \text{rk}(y) = \text{rk}(v \lor y)$, as claimed.

Now substitute $\mu(v)v \to \mu(v)t^{\text{rk}(z)-\text{rk}(v)}$ and $\mu(y)y \to \mu(y)t^{n-\text{rk}(y)-\text{rk}(z)}$ in the right-hand side of equation (33). Then by Claim 2 we have
\[
v y \to t^{n-\text{rk}(v)-\text{rk}(y)} = t^{n-\text{rk}(v \lor y)}.
\]
Now $v \lor y$ is just $vy$ in the Möbius algebra $A(L)$. Hence if we further substitute $\mu(x)x \to \mu(x)t^{n-\text{rk}(x)}$ in the left-hand side of (33), then the product will be preserved. We thus obtain
\[
\sum_{x \in L} \mu(x)t^{n-\text{rk}(x)} = \left( \sum_{v \leq z} \mu(v)t^{\text{rk}(z)-\text{rk}(v)} \right) \left( \sum_{y \land z = 0} \mu(y)t^{n-\text{rk}(y)-\text{rk}(z)} \right),
\]
as desired. \hfill \Box

Corollary 4.8. Let $L$ be a geometric lattice of rank $n$ and $a$ an atom of $L$. Then
\[
\chi_L(t) = (t-1) \sum_{y \land a = 0} \mu(y)t^{n-1-\text{rk}(y)}.
\]

Proof. The atom $a$ is modular (Example 4.9(b)), and $\chi_a(t) = t-1$. \hfill \Box

Corollary 4.8 provides a nice context for understanding the operation of coning defined in Chapter 1, in particular, Exercise 2.1. Recall that if $A$ is an affine arrangement in $K^n$ given by the equations
\[
L_1(x) = a_1, \ldots, L_m(x) = a_m,
\]
then the cone $xA$ is the arrangement in $K^n \times K$ (where $y$ denotes the last coordinate) with equations
\[
L_1(x) = a_1y, \ldots, L_m(x) = a_my, \ y = 0.
\]
Let $H_0$ denote the hyperplane $y = 0$. It is easy to see by elementary linear algebra that

$$L(A) \cong L(cA) - \{ x \in L(A) : x \geq H_0 \} = L(A) - L(A^{H_0}).$$

Now $H_0$ is a modular element of $L(A)$ (since it’s an atom), so Corollary 4.8 yields

$$\chi_{cA}(t) = (t - 1) \sum_{y \geq H_0} \mu(y)t^{(n+1) - 1 - \text{rk}(y)}$$

$$= (t - 1)\chi_A(t).$$

There is a left inverse to the operation of coning. Let $A$ be a nonempty linear arrangement in $K^{n+1}$. Let $H_0 \in A$. Choose coordinates $(x_0, x_1, \ldots, x_n)$ in $K^{n+1}$ so that $H_0 = \ker(x_0)$. Let $A$ be defined by the equations

$$x_0 = 0, \quad L_1(x_0, \ldots, x_n) = 0, \ldots, \quad L_m(x_0, \ldots, x_n) = 0.$$

Define the \textit{deconing} $c^{-1}A$ (with respect to $H_0$) in $K^n$ by the equations

$$L_1(1, x_1, \ldots, x_n) = 0, \ldots \quad L_m(1, x_1, \ldots, x_n) = 0.$$

Clearly $c(c^{-1}A) = A$ and $L(c^{-1}A) \cong L(A) - \{ x \in L(A) : x \geq H_0 \}$.

### 4.3. Supersolvable lattices

For some geometric lattices $L$, there are “enough” modular elements to give a factorization of $\chi_L(t)$ into linear factors.

**Definition 4.13.** A geometric lattice $L$ is \textit{supersolvable} if there exists a modular maximal chain, i.e., a maximal chain $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ such that each $x_i$ is modular. A central arrangement $A$ is \textit{supersolvable} if its intersection lattice $L_A$ is supersolvable.

**Note.** Let $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ be a modular maximal chain of the geometric lattice $L$. Clearly then each $x_{i-1}$ is a modular element of the interval $[0, x_i]$. The converse follows from Proposition 4.10(b): if $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ is a maximal chain for which each $x_{i-1}$ is modular in $[0, x_i]$, then each $x_i$ is modular in $L$.

**Note.** The term “supersolvable” comes from group theory. A finite group $\Gamma$ is \textit{supersolvable} if and only if its subgroup lattice contains a maximal chain all of whose elements are normal subgroups of $\Gamma$. Normal subgroups are “nice” analogues of modular elements; see [17, Example 2.5] for further details.

**Corollary 4.9.** Let $L$ be a supersolvable geometric lattice of rank $n$, with modular maximal chain $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$. Let $T$ denote the set of atoms of $L$, and set

$$e_i = \# \{ a \in T : a \leq x_i, \ a \nleq x_{i-1} \}.$$  \hspace{1cm} (34)

Then $\chi_L(t) = (t - e_1)(t - e_2)\cdots(t - e_n)$.

**Proof.** Since $x_{n-1}$ is modular, we have

$$y \land x_{n-1} = \hat{0} \Leftrightarrow y \in T \text{ and } y \nleq x_{n-1}, \text{ or } y = \hat{0}.$$
By Theorem 4.13 we therefore have

\[ \chi_L(t) = \chi_{x_{n-1}}(t) \left[ \sum_{q \in T} \mu(a) t^{n-rk(a)-rk(x_{n-1})} + \mu(\hat{0}) t^{n-rk(\hat{0})-rk(x_{n-1})} \right]. \]

Since \( \mu(a) = -1, \mu(\hat{0}) = 1, \) \( rk(a) = 1, \) \( rk(\hat{0}) = 0, \) and \( rk(x_{n-1}) = n - 1, \) the expression in brackets is just \( t - e_n. \) Now continue this with \( L \) replaced by \([\hat{0}, x_{n-1}]\) (or use induction on \( n \)).

**Note.** The positive integers \( e_1, \ldots, e_n \) of Corollary 4.9 are called the *exponents* of \( L. \)

**Example 4.11.**

(a) Let \( L = B_n, \) the boolean algebra of rank \( n. \) By Example 4.9(d) every element of \( B_n \) is modular. Hence \( B_n \) is supersolvable. Clearly each \( e_i = 1, \) so \( \chi_{B_n}(t) = (t - 1)^n. \)

(b) Let \( L = B_n(q), \) the lattice of subspaces of \( \mathbb{F}_q^n. \) By Example 4.9(e) every element of \( B_n(q) \) is modular, so \( B_n(q) \) is supersolvable. If \( \binom{k}{n} \) denotes the number of \( j \)-dimensional subspaces of a \( k \)-dimensional vector space over \( \mathbb{F}_q, \) then

\[ e_i = \binom{i}{1} - \binom{i-1}{1} = \frac{q^i - 1}{q - 1} - \frac{q^{i-1} - 1}{q - 1} = q^{i-1}. \]

Hence

\[ \chi_{B_n(q)}(t) = (t - 1)(t - q)(t - q^2) \cdots (t - q^{n-1}). \]

In particular, setting \( t = 0 \) gives

\[ \mu_{B_n(q)}(\hat{1}) = (-1)^n q_{\binom{2}{n}}. \]

**Note.** The expression \( \binom{k}{n} \) is called a *q-binomial coefficient.* It is a polynomial in \( q \) with many interesting properties. For the most basic properties, see e.g. [18, pp. 27–30].

(c) Let \( L = \Pi_n, \) the lattice of partitions of the set \([n]\) (a geometric lattice of rank \( n - 1. \)) By Proposition 4.9, a maximal chain of \( \Pi_n \) is modular if and only if it has the form \( \hat{0} = \pi_0 < \pi_1 < \cdots < \pi_{n-1} = 1, \) where \( \pi_i \) for \( i > 0 \) has exactly one nonsingleton block \( B_i \) (necessarily with \( i + 1 \) elements), with \( B_1 \subseteq B_2 \cdots \subseteq B_{n-1} = [n]. \) In particular, \( \Pi_n \) is supersolvable and has exactly \( n! / 2 \) modular chains for \( n > 1. \) The atoms covered by \( \pi_i \) are the partitions with one nonsingleton block \( \{ j, k \} \subseteq B_i. \) Hence \( \pi_i \) lies above exactly \( \binom{i+1}{2} \) atoms, so

\[ e_i = \binom{i+1}{2} - \binom{i}{2} = i. \]

It follows that \( \chi_{\Pi_n}(t) = (t - 1)(t - 2) \cdots (t - n + 1) \) and \( \mu_{\Pi_n}(\hat{1}) = (-1)^{n-1}(n - 1)!. \) Compare Corollary 2.2. The polynomials \( \chi_{B_n}(t) \) and \( \chi_{\Pi_n}(t) \) differ by a factor of \( t \) because \( B_n(t) \) is an arrangement in \( K^n \) of
rank $n - 1$. In general, if $\mathcal{A}$ is an arrangement and $\text{ess}(\mathcal{A})$ its essentialization, then
\begin{equation}
\tau^{\text{rk}(\text{ess}(\mathcal{A}))} \chi_{\mathcal{A}}(t) = \tau^{\text{rk}(\mathcal{A})} \chi_{\text{ess}(\mathcal{A})}(t).
\end{equation}

(See Lecture 1, Exercise 2.)

**Note.** It is natural to ask whether there is a more general class of geometric lattices $L$ than the supersolvable ones for which $\chi_L(t)$ factors into linear factors (over $\mathbb{Z}$). There is a profound such generalization due to Terao [22] when $L$ is an intersection poset of a linear arrangement $\mathcal{A}$ in $K^n$. Write $K[x] = K[x_1, \ldots, x_n]$ and define
\[
\mathcal{T}(A) = \{(p_1, \ldots, p_n) \in K[x]^n : p_i(H) \subseteq H \text{ for all } H \in \mathcal{A}\}.
\]
Here we are regarding $(p_1, \ldots, p_n) : K^n \to K^n$, viz., if $(a_1, \ldots, a_n) \in K^n$, then
\[
(p_1, \ldots, p_n)(a_1, \ldots, a_n) = (p_1(a_1, \ldots, a_n), \ldots, p_n(a_1, \ldots, a_n)).
\]
The $K[x]$-module structure $K[x] \times \mathcal{T}(A) \to \mathcal{T}(A)$ is given explicitly by
\[
q \cdot (p_1, \ldots, p_n) = (q p_1, \ldots, q p_n).
\]

Note, for instance, that we always have $(x_1, \ldots, x_n) \in \mathcal{T}(A)$. Since $\mathcal{A}$ is a linear arrangement, $\mathcal{T}(A)$ is indeed a $K[x]$-module. (We have given the most intuitive definition of the module $\mathcal{T}(A)$, though it isn’t the most useful definition for proofs.) It is easy to see that $\mathcal{T}(A)$ has rank $n$ as a $K[x]$-module, i.e., $\mathcal{T}(A)$ contains $n$, but not $n + 1$, elements that are linearly independent over $K[x]$. We say that $\mathcal{A}$ is a free arrangement if $\mathcal{T}(A)$ is a free $K[x]$-module, i.e., there exist $Q_1, \ldots, Q_n \in \mathcal{T}(A)$ such that every element $Q \in \mathcal{T}(A)$ can be uniquely written in the form $Q = q_1 Q_1 + \cdots + q_n Q_n$, where $q_i \in K[x]$. It is easy to see that if $\mathcal{T}(A)$ is free, then the basis $\{Q_1, \ldots, Q_n\}$ can be chosen to be homogeneous, i.e., all coordinates of each $Q_i$ are homogeneous polynomials of the same degree $d_i$. We then write $d_i = \deg Q_i$. It can be shown that supersolvable arrangements are free, but there are also nonsupersolvable free arrangements. The property of freeness seems quite subtle; indeed, it is unknown whether freeness is a matroidal property, i.e., depends only on the intersection lattice $L_{\mathcal{A}}$ (regarding the ground field $K$ as fixed). The remarkable “factorization theorem” of Terao is the following.

**Theorem 4.14.** Suppose that $\mathcal{T}(A)$ is free with homogeneous basis $Q_1, \ldots, Q_n$. If $\deg Q_i = d_i$ then
\[
\chi_{\mathcal{A}}(t) = (t - d_1)(t - d_2) \cdots (t - d_n).
\]

We will not prove Theorem 4.14 here. A good reference for this subject is [13, Ch. 4].

Returning to supersolvability, we can try to characterize the supersolvable property for various classes of geometric lattices. Let us consider the case of the bond lattice $L_G$ of the graph $G$. A graph $H$ with at least one edge is doubly connected if it is connected and remains connected upon the removal of any vertex (and all incident edges). A maximal doubly connected subgraph of a graph $G$ is called a block of $G$. For instance, if $G$ is a forest then its blocks are its edges. Two different blocks of $G$ intersect in at most one vertex. Figure 5 shows a graph with eight blocks, five of which consist of a single edge. The following proposition is straightforward to prove (Exercise 16).
Proposition 4.11. Let $G$ be a graph with blocks $G_1, \ldots, G_k$. Then

$$L_G \cong L_{G_1} \times \cdots \times L_{G_k}.$$ 

It is also easy to see that if $L_1$ and $L_2$ are geometric lattices, then $L_1$ and $L_2$ are supersolvable if and only if $L_1 \times L_2$ is supersolvable (Exercise 18). Hence in characterizing supersolvable graphs $G$ (i.e., graphs whose bond lattice $L_G$ is supersolvable) we may assume that $G$ is doubly connected. Note that for any connected (and hence a fortiori doubly connected) graph $G$, any coatom $\pi$ of $L_G$ has exactly two blocks.

Proposition 4.12. Let $G$ be a doubly connected graph, and let $\pi = \{A, B\}$ be a coatom of the bond lattice $L_G$, where $\#A \leq \#B$. Then $\pi$ is a modular element of $L_G$ if and only if $\#A = 1$, say $A = \{v\}$, and the neighborhood $N(v)$ (the set of vertices adjacent to $v$) forms a clique (i.e., any two distinct vertices of $N(v)$ are adjacent).

**Proof.** The proof parallels that of Proposition 4.9, which is a special case. Suppose that $\#A > 1$. Since $G$ is doubly connected, there exist $u, v \in A$ and $u', v' \in B$ such that $u \neq v, u' \neq v', uu' \in E(G)$, and $vv' \in E(G)$. Set $\sigma = \{(A \cup u') - v, (B \cup v') - u\}$. If $G$ has $n$ vertices then $\text{rk}(\pi) = \text{rk}(\sigma) = n - 2$, $\text{rk}(\pi \vee \sigma) = n - 1$, and $\text{rk}(\pi \wedge \sigma) = n - 4$. Hence $\pi$ is not modular.

Assume then that $A = \{v\}$. Suppose that $av, bv \in E(G)$ but $ab \notin E(G)$. We need to show that $\pi$ is not modular. Let $\sigma = \{A - \{a, b\}, \{a, b, v\}\}$. Then

$$\sigma \vee \pi = \hat{1}, \quad \sigma \wedge \pi = \{A - \{a, b\}, a, b, v\}$$

$$\text{rk}(\sigma) = \text{rk}(\pi) = n - 2, \quad \text{rk}(\sigma \vee \pi) = n - 1, \quad \text{rk}(\sigma \wedge \pi) = n - 4.$$ 

Hence $\pi$ is not modular.

Conversely, let $\pi = \{A, v\}$. Assume that if $av, bv \in E(G)$ then $ab \in E(G)$. It is then straightforward to show (Exercise 8) that $\pi$ is modular, completing the proof.

As an immediate consequence of Propositions 4.10(b) and 4.12 we obtain a characterization of supersolvable graphs.

**Corollary 4.10.** A graph $G$ is supersolvable if and only if there exists an ordering $v_1, v_2, \ldots, v_n$ of its vertices such that if $i < k$, $j < k$, $v_i v_k \in E(G)$ and $v_j v_k \in E(G)$,
then $v_iv_j \in E(G)$. Equivalently, in the restriction of $G$ to the vertices $v_1, v_2, \ldots, v_i$, the neighborhood of $v_i$ is a clique.

Note. Supersolvable graphs $G$ had appeared earlier in the literature under the names chordal, rigid circuit, or triangulated graphs. One of their many characterizations is that any circuit of length at least four contains a chord. Equivalently, no induced subgraph of $G$ is a $k$-cycle for $k \geq 4$. 