Thm (E-L): Let \( \mathcal{F} \) be \( k \)-uniform, intersecting, and \( |\mathcal{F}| > k^k \). Then \( \mathcal{F} \) is 2-colorable.

Proof: By contradiction, \( \mathcal{F} \) not 2-colorable \( \Rightarrow |\mathcal{F}| \leq k^k \)

Let \( d(B) = \# S \in \mathcal{F} \) s.t. \( B \subseteq S \).

Claim: \( \exists \) a sequence \( x_1, \ldots, x_k \in \bigcap \mathcal{F} \) s.t.
\[
d(B_i) \geq \frac{|\mathcal{F}|}{k^k} \quad B_i = \{x_1, \ldots, x_i\}
\]

Claim \( \Rightarrow \) thm: \( 1 \geq d(B) \geq \frac{|\mathcal{F}|}{k^k} \)

Proof of claim by induction on \( i \):

Base: \( i = 1 \) pigeonhole

Step of Induction: Suppose \( B_i \) for \( i \) already chosen.

If \( B_i \cap S_j \neq \emptyset \) for \( S_j \in \mathcal{F} \), can 2-color \( \mathcal{F} \)

So let \( x_i \in S_j \) be the one which belongs to the largest \# of subsets containing \( B_i \).

This is \( \geq d(B) / k \) by sim. argument to base.

"Okay, so, I'm going to move to colorings of graphs."

Notation: Let \( d(G) = \max_{v \in G} d(v) \)
\[
\chi(G) = \text{usu.}
\]
Proposition

\[ d(G) = k \Rightarrow \chi(G) \leq k+1 \]

Prop'sn: 

\[ d(G) = k \text{, } \exists x \in V(G) \text{ s.t. } d(x) = k \Rightarrow \]

pf: Let small deg vertex be root, orient edges acyclicly.

sort out vertices by distance from \( x \), note all edges are w/in same dist or from \( x \).

Run greedy algorithm starting \( x \). Note that always have one uncolored neighbor until \( x + d(x) = k-1 \).

(similar in Bollobás) (aha!)

(This is a real) Theorem (Brooks 1949)

\[ d(G) \leq k \text{ and } G \neq K_{k+1} \text{ and } k \geq 3. \]

Then \( \chi(G) \leq k \). (G connected)

pf: Need only prove for \( G \) being \( k \)-regular, as above.

Def'n: \( G \) is \( k \)-connected if \( G - \{x_1, \ldots, x_k\} \) is connected.

Okay, for \( G \) 2-connected, pinch vertex \( \checkmark \)

\( G \) 3-connected. Take \( x_n \), let \( x_1, x_2 \in N(x_n) \). 

\( \exists x_1, x_2 \in E(G) \). Look at \( G - x_1, x_2 \). If \( G \) 3-connected, still connected. Construct sequence w/ 

\( x_n \) as root, forward neighbor rule, \( x_1 \) and \( x_2 \)

1st 2 vertices + color them the same.

color \( \Rightarrow \checkmark \) cool.