Two more applications of linear algebra complete

1) Packing with bipartite graphs
   \[ E(G) = \bigcup \mathcal{H}_{i} \text{, "edge decomposition"} \] and \( i = 1 \) s.t., \( \mathcal{H}_i = K_{r,q} \)

Question: What is \( m_i \) or \( r \)?

**Thm** \( r(K_n) \geq n-1 \)

**Pf** Let \( M = \text{adjacency matrix of } G = K_n \)
\[
A = \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \end{pmatrix}
\]

Where the star means lower triangle is all 0's.
Then \( rk(M) = n-1 \) and \( rk(A_i) = 1 \) and
\[
rk(A_1 + \cdots + A_r) \leq rk(A_1) + \cdots + rk(A_r) = r \Rightarrow r \geq n-1
\]

2) CS Matrix product testing
   \( A, B \in GL(n, q) \), non-matrices over \( \mathbb{F} \)

\[ \exists \text{ Probabilistic algorithm testing if } A \otimes B = A \cdot B \]

in \( O(n^2) \) steps (w/ high prob)

**Given** \( A, B, A \otimes B \), let
\[
v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \end{pmatrix}, c \in \mathbb{F}, v \text{ random}
\]

Checking \( (A \otimes B) v \neq A (B \cdot v) \) takes \( O(n^2) \)

**Lemma:** If \( A \otimes B \neq A \cdot B \), then single check
will fail w/ prob \( \geq 1/2 \) (actually \( 1 - 1/q \))

That's because the subspace of \( v \) s.t. \( A \otimes B v = A B v \)
is \( v \) s.t. \( (A \otimes B - A \cdot B) v = 0 \), which must have
dimension at least \( r \), where most \( \frac{r}{n} \) are missing, it is \( 1 - \frac{r}{n} \)
so prob \( \approx \frac{1}{2} \cdot \frac{n}{r} \)
Prob gives wrong answer to \( V \), iterate...

**Puzzle**

K_{10} packing w/ Peterson graphs

It's impossible! Pf by linear algebra (kidding)

(look at eigenvalues of adjacency matrices... )