\[
\beta_n = \text{poset of } P(\mathbb{N})
\]

**Thm (Sperner, 1928)**

Every antichain has size \( \leq \left( \begin{array}{c} n \\ \frac{n}{2} \end{array} \right) \)

**Pf (from Jukna):** \( \mathcal{F} = \{ A : A \subseteq [n] \text{ antichain} \}

Pick \( A \in \mathcal{F} \) and look at all max. chains in \([n]\)
containing \( A \). \( |A| = k \Rightarrow \# \text{ chains } = \binom{n}{n-k} \)
Also none of the max. chains in \( B \), meet \( \mathcal{F} \) more than once, so
\[ n! \geq \left| \bigcup (n-|A|)! \geq \frac{n!}{n^{|A|}} \cdot \left( \frac{n}{|A|} \right)^{|A|} \]
\[ \Rightarrow |B| \leq \left( \frac{n}{|A|} \right)^{|A|} \]

**Thm (Mahtel, 1907)**

\[ |V(G)| = 2n \quad |E(G)| \geq n^2 + 1 \]

\[ \Rightarrow \Delta \leq G \]

**Pf:** (From Jukna Ch 4) induction base \( n=1 \)

**Step:** 2n+2 vertices, pick \( (x,y) \in E \) \( H = G - x - y \)

\[ |E(H)| \geq n^2 + 1 \Rightarrow \sqrt{|E(H)|} < n^2 + 1 \Rightarrow \]

\[ |N(x)| + |N(y)| \geq 2n + 1 \Rightarrow N(x) \cap N(y) \neq \emptyset \]

Second proof is more lively:

\( A \subset V(G) \) largest indep set (i.e. \( A \text{indep, } |A| = \alpha(G) \))

\[ B = V - A \quad \forall x \in B \quad N(x) \cap A \neq \emptyset \text{ and } \]

\[ |N(x)| \leq \alpha(G) = |A| \quad \text{co/wh \exists } x \in B \text{ s.t. } u \in x \wedge v \in y \]

\[ |E| \leq \sum_{x \in B} \deg x \leq \sum |A| = |B| \cdot |A| \leq \left( \frac{|A| + |B|}{2} \right)^2 = \frac{n^2}{2} \]

**Thm (Graham-Kleitman 1973)**

\( \alpha : E(K_n) \leftrightarrow \Sigma(9) \) labelling of edges

Then the longest increasing trail has length \( > (n-1) \)

(trail: oriented path that can go through the same \( v \) more than once)
Proof: Let $w_* = \text{length of longest inc. tail ending at } x$

$\text{ETST } \leq w_* \geq n(n-1)$

Start: $x \leq w = 0$

$i^{th}$ step: look at $v_i = x y$

this increases $w$ by $y$ makes

$w' = \max \{ w, w + 1 \}$

$\Rightarrow$ every edge addition increases weight sum by 2

$\Rightarrow w_* \geq (\frac{3}{2})2$