Introduction to tilings

Region \( R \), tiles (a)

Question 1. Given \( T = T_1, T_2, \ldots \), set of tiles, is \( R \) tileable?

2. How many ways, etc.

- First piece of trivia
  - Dominoes: \( 2 \times n \) tile is Fibonacci
  - Checkerboard (not sufficient)

3. dom tilings \( \iff \) perfect matching of duals, which can be done in poly time \( \Rightarrow \) ?
   in poly time \( \langle \text{hop} \Rightarrow O(1^{11111}) \rangle \)

Thm (stated vault, proved by Thurston)

\( \Gamma \) simp conn \( \Rightarrow \) lin time alg to determine tilability of \( \Gamma \)

by dominoes \( \langle \text{lin w.r.t. area} \rangle \)

Thm (Thurston)

If \( \Gamma \) is simply connected, every two domino tilings are connected by a sequence of \( 2 \times 2 \) moves

\( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \)

Pf: Fix a pt \( O \) in \( \Gamma \). Define \( f: \partial \Gamma \rightarrow \mathbb{Z} \)

height \( f \in \mathbb{Z} \), \( f(x) = \text{height} \)

(picture underlying chessboard + look at color of each square)
Claim: $\Gamma$ s.c., tileable $\Rightarrow$ f well defined.

Pf: By induction, remove a on $\Gamma$ (not just $\partial \Gamma$) domino, check around that domino.$\checkmark$

Lemma: $\Gamma$ s.c., $\Gamma = \sqcup_i R_i \Rightarrow R_i$ s.c.$\Rightarrow$

$\exists i \in \mathbb{K}$ s.t. $R_i$ is s.c.

(use lemma to show you can remove domino in pf)

(Ex: false in $\mathbb{R}^3$)

Define poset $\mathcal{P}_f$ on all domino tilings of $\Gamma$/$f \leq f'$ if $\forall$ pt in $\Gamma$ $f(x) \leq f'(x)$ height $f$'s

Lemma: $\exists \text{ max}, \exists \text{ min} \in \mathcal{P}_f$

Pf of lemma: Lemma. $\$ local min $\Rightarrow \max$ of $f$ on $\partial \Gamma$

$\Rightarrow \exists \text{ max}$

Pf: Suppose not, then $\exists$ s.t. (since $\exists \text{ max}$) must have*

Flipping that square gives less $\exists$

This lemma of lemma allows us to prove the whole theorem.

Algorithm: Calculate $h$ on $\partial \Gamma$ (well defined if tileable). Find $\max$ $\exists$ must have domino $\square$, remove it, repeat.

This gives local (hence global) min or max.

Furthermore, can keep 2x2 flipping to decrease $h$'s until reach global min.