Instructions: Solve your favourite problems from the list below. Open problems are marked with \( \star \); hard (but feasible) problems are marked with \( \star \).

1. Let \( a(P) \) denote the number of angles determined by ordered triples of a set \( P \) of non-collinear points in the plane. (We count angles \( 0^\circ \leq \angle(p_1,p_2,p_3) < 180^\circ \).) For \( n \in \mathbb{N}, n \geq 3 \), let \( a(n) = \min_{|P|=n} a(P) \).

   (a) For every \( n \in \mathbb{N}, n \geq 3 \), find a set \( P_n \) of \( n \) points such that \( a(P_n) = n - 2 \).

   (b) Show that \( a(n) = \Omega(n) \).

   (c) Prove or disprove that \( a(n) = n - 2 \). \( \star \)

2. Prove the Sylvester-Gallai Theorem for a system of non-concurrent \textit{pseudo-lines} in the plane: You are given a set of curves in the plane such that any two curves intersect in exactly one point, and no point is incident to all the curves. Show that there is a point incident to exactly two curves.

3. We are given \( n \) points and \( \ell \) curves or surfaces in \( \mathbb{R}^d \), \( d \geq 2 \). For any two real numbers \( a, b > 1 \), find two reals \( e, f \in \mathbb{R} \) (in terms of \( a \) and \( b \)) such that

   \[
   \#(k\text{-rich lines}) = O\left( \frac{n^a}{k^b} \right), \quad \forall k \in \mathbb{N} \iff \#(\text{incidences}) = O(n^e \ell^f).
   \]

4. (Elekes) Let \( X \) and \( Y \) be two sets of \( n \) real numbers \( (X,Y \subset \mathbb{R}, |X| = |Y| = n) \). Consider the Cartesian product \( P = X \times Y = \{(x,y) \in \mathbb{R}^2 : x \in X, y \in Y \} \) in the plane. Show that the number of collinear triples of \( P \) is at most \( O(n^4 \log n) \).

5. (Erdős) Consider \( n \) points in an integer lattice section \( P = \{(a,b) \in \mathbb{N}^2 : 1 \leq a \leq \sqrt{n}, 1 \leq b \leq \sqrt{n} \} \) in the plane. Show that for every \( \ell \in \mathbb{N}, \ell \geq \sqrt{n} \), there are \( \ell \) lines in the plane such that the number of incidences with \( P \) is at least \( \Omega(n^{2/3} \ell^{2/3} + n + \ell) \).

6. (Valtr, 2005) Let \( \partial B \) denote the boundary curve of a convex compact body \( B \). Find a strictly convex compact body \( B \) in the plane with the following property: There are \( n \) translates of \( \partial B \) and \( n \) points in the plane such that the number of point-curve incidences is at least \( \Omega(n^{4/3}) \). \( \star \)

7. We are given \( n \) points and \( \ell \) circles in the plane such that exactly \( x \) pairs of circles intersect. Show that the number of point-circle incidences is at most \( O(n^{2/3}x^{1/3} + n + \ell) \).

8. Given a set \( S_n \) of \( n \) points in the plane, let \( g(S_n) \) denote the number of \textit{unit peremeter triangles} (that is, triangles where the sum of the three edge lengths is one).

   (a) Show that \( g(S_n) = O(n^{7/3}) \).

   (b) Show that \( g(S_n) = O(n^{16/7}) \). \( \star \)

9. (Hanani, 1934) If any two edges of a topological graph cross an even number of times, then the graph is planar. \( \star \)