1. (a) Prove that the maximum length of the Davenport-Schinzel sequence of order 2 over an alphabet of $n$ letters is $\lambda_2(n) = 2n - 1$. $\otimes$
(b) Show that for every $n$ and $s$, $\lambda_s(n) \leq 1 + (s + 1)\binom{n}{2}$. $\otimes$
(c) Show that the lower envelope of $n$ half-lines in the plane has $O(n)$ complexity. $\otimes$

2. Let $P_1, P_2, \ldots, P_m$ be convex polygons in the plane such that their vertex sets are disjoint (but the polygons are not necessarily disjoint). Assume that there are a total of $n$ vertices and they are in general position. Show that the number of lines intersecting all polygons and tangent to exactly two of them is $O(\lambda_3(n))$. $\otimes$

3. Consider a cell $C$ in an arrangement of $n$ line segments in the plane. Let $|C|$ denote the complexity of the boundary of $C$.
(a) Show that $|C| = O(\lambda_4(n))$. $\otimes$ (b) Show that $|C| = O(\lambda_3(n))$. $\star \otimes$

4. You are given a function $\psi : \mathbb{N}^2 \to \mathbb{N}$ that satisfies $\psi(2, n) = 2n$ and the property that for every $p \in \mathbb{N}$, $1 \leq p \leq m$, there are $n_1, n_2 \in \mathbb{N}$ such that $n = n_1 + n_2$ and

$$\psi(m, n) \leq 4m + 4n_2 + \psi(p, n_2) + p \cdot \psi\left(\left\lfloor m/p \right\rfloor, \left\lfloor n_1/p \right\rfloor\right).$$

(a) Prove that $\psi(2^j, n) \leq 4j \cdot 2^j + 6n$, for $j \geq 1$. (b) Show $\psi(2n, n) = O(n \log^* n)$. $\star$

5. Let $\text{ex}(n, M)$ denote the maximum number of 1 entries in an $n \times n$ size 0-1 matrix that does not contain as submatrix any matrix of the family $M$. (a) (Füredi-Hajnal, 1992) $\text{ex}(n, A) = \lambda_3(n) + O(n)$. $\star$ (b) (Füredi, Bienstock-Győri, 1990) $\text{ex}(n, B) = \Theta(n \log n)$.

$$A = \begin{pmatrix} 1 & * & 1 & * \\ * & 1 & * & 1 \end{pmatrix}, \quad B = \begin{pmatrix} * & 1 & 1 \\ 1 & * & 1 \end{pmatrix}.$$

6. (a) Count the edges in the arrangement of $n$ planes in general position in $\mathbb{R}^3$. $\otimes$
(b) Express the number of $k$-dimensional faces in an arrangement of $n$ hyperplanes in general position in $\mathbb{R}^d$ in terms of $d$, $k$, and $n$. $\otimes$
(c) Prove that for every fixed $d \in \mathbb{N}$ the number of unbounded cells in an arrangement of $n$ hyperplanes is $O(n^{d-1})$. $\otimes$

7. (Shrivastava) In the arrangement of $n$ lines in the plane, consider the $k$-level and the $(2k)$-level for some $k \in \mathbb{N}$, $1 \leq k \leq n/2$. Show that there is a curve along the lines of the arrangement that separates these two levels and consists of $O(n/k)$ line segments.

8. (Sharir, 2001) Consider an arrangement of $n$ lines in the plane. Let $S$ be the largest subset of vertices such that none of the lines passes below more than $k$ points of $S$.
(a) Show that $|S| = O(n\sqrt{k})$. (b) Find an arrangement where $|S| = \Omega(n\sqrt{k})$.
(c) For $n$ points in general position in the plane, $C$ is a set of circles such that each circle passes through three points and contains at most $k$ points. Show that $|C| = O(nk^{2/3})$. 