Appendix B

Finite difference methods for wave equations

Many types of numerical methods exist for computing solutions to wave equations – finite differences are the simplest, though often not the most accurate ones.

Consider for illustration the 1D time-dependent problem

\[ m(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in [0, 1], \]

with smooth \( f(x,t) \), and, say, zero initial conditions. The simplest finite difference scheme for this equation is set up as follows:

- Space is discretized over \( N + 1 \) points as \( x_j = j\Delta x \) with \( \Delta x = \frac{1}{N} \) and \( j = 0, \ldots, N \).

- Time is discretized as \( t_n = n\Delta t \) with \( n = 0, 1, 2, \ldots \). Call \( u_j^n \) the computed approximation to \( u(x_j, t_n) \). (In this appendix, \( n \) is a superscript.)

- The centered finite difference formula for the second-order spatial derivative is

\[
\frac{\partial^2 u}{\partial x^2}(x_j, t_n) = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + O((\Delta x)^2),
\]

provided \( u \) is sufficiently smooth – the \( O(\cdot) \) notation hides a multiplicative constant proportional to \( \partial^4 u/\partial x^4 \).
• Similarly, the centered finite difference formula for the second-order
time derivative is

\[ \frac{\partial^2 u}{\partial t^2}(x_j, t_n) = \frac{u_{j}^{n+1} - 2u_{j}^{n} + u_{j}^{n-1}}{(\Delta t)^2} + O((\Delta t)^2), \]

provided \( u \) is sufficiently smooth.

• Multiplication by \( m(x) \) is realized by multiplication on the grid by
\( m(x_j) \). Gather all the discrete operators to get the discrete wave equa-
tion.

• The wave equation is then solved by marching: assume that the values
of \( u_{j}^{n-1} \) and \( u_{j}^{n} \) are known for all \( j \), then isolate \( u_{j}^{n+1} \) in the expression
of the discrete wave equation.

Dirichlet boundary conditions are implemented by fixing, e.g., \( u_0 = a \).
Neumann conditions involve a finite difference, such as \( \frac{u_{1} - u_{0}}{\Delta x} = a \). The more
accurate, centered difference approximation \( \frac{u_{1} - u_{-1}}{2\Delta x} = a \) with a ghost node at
\( u_{-1} \) can also be used, provided the discrete wave equation is evaluated one
more time at \( x_0 \) to close the resulting system. In 1D the absorbing boundary
condition has the explicit form \( \frac{1}{v} \partial_t u \pm \partial_x u = 0 \) for left (-) and right-going (+)
waves respectively, and can be implemented with adequate differences (such
as upwind in space and forward in time).

The grid spacing \( \Delta x \) is typically chosen as a small fraction of the rep-
resentative wavelength in the solution. The time step \( \Delta t \) is limited by the
CFL condition \( \Delta t \leq \Delta x / \max_x c(x) \), and is typically taken to be a fraction
thereof.

In two spatial dimensions, the simplest discrete Laplacian is the 5-point
stencil which combines the two 3-point centered schemes in \( x \) and in \( y \). Its
accuracy is also \( O(\max\{\Delta x, \Delta y\}) \). Designing good absorbing boundary
conditions is a somewhat difficult problem that has a long history. The
currently most popular solution to this problem is to slightly expand the
computational domain using an absorbing, perfectly-matched layer (PML).

More accurate schemes can be obtained from higher-order finite differ-
ences. Low-order schemes such as the one explained above typically suffer
from unacceptable numerical dispersion at large times. If accuracy is a big
concern, spectral methods (spectral elements, Chebyshev polynomials, etc.)
are by far the best way to solve wave equations numerically with a controlled,
small number of points per wavelength.
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