Appendix B

Finite difference methods for wave equations

Many types of numerical methods exist for computing solutions to wave equations – finite differences are the simplest, though often not the most accurate ones.

Consider for illustration the 1D time-dependent problem

$$m(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad x \in [0, 1],$$

with smooth $f(x, t)$, and, say, zero initial conditions. The simplest finite difference scheme for this equation is set up as follows:

- Space is discretized over $N + 1$ points as $x_j = j\Delta x$ with $\Delta x = \frac{1}{N}$ and $j = 0, \ldots, N$.

- Time is discretized as $t_n = n\Delta t$ with $n = 0, 1, 2, \ldots$. Call $u_j^n$ the computed approximation to $u(x_j, t_n)$. (In this appendix, $n$ is a superscript.)

- The centered finite difference formula for the second-order spatial derivative is

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_n) = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} + O((\Delta x)^2),$$

provided $u$ is sufficiently smooth – the $O(\cdot)$ notation hides a multiplicative constant proportional to $\partial^4 u / \partial x^4$. 

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Similarly, the centered finite difference formula for the second-order time derivative is
\[
\frac{\partial^2 u}{\partial t^2}(x_j, t_n) = \frac{u_{j}^{n+1} - 2u_{j}^{n} + u_{j}^{n-1}}{(\Delta t)^2} + O((\Delta t)^2),
\]
provided \(u\) is sufficiently smooth.

- Multiplication by \(m(x)\) is realized by multiplication on the grid by \(m(x_j)\). Gather all the discrete operators to get the discrete wave equation.

- The wave equation is then solved by marching: assume that the values of \(u_{j}^{n-1}\) and \(u_{j}^{n}\) are known for all \(j\), then isolate \(u_{j}^{n+1}\) in the expression of the discrete wave equation.

Dirichlet boundary conditions are implemented by fixing, e.g., \(u_0 = a\). Neumann conditions involve a finite difference, such as \(\frac{u_{1} - u_{0}}{\Delta x} = a\). The more accurate, centered difference approximation \(\frac{u_{1} - u_{-1}}{2\Delta x} = a\) with a ghost node at \(u_{-1}\) can also be used, provided the discrete wave equation is evaluated one more time at \(x_0\) to close the resulting system. In 1D the absorbing boundary condition has the explicit form \(\frac{1}{c} \partial_t u \pm \partial_x u = 0\) for left (-) and right-going (+) waves respectively, and can be implemented with adequate differences (such as upwind in space and forward in time).

The grid spacing \(\Delta x\) is typically chosen as a small fraction of the representative wavelength in the solution. The time step \(\Delta t\) is limited by the CFL condition \(\Delta t \leq \Delta x / \max_x c(x)\), and is typically taken to be a fraction thereof.

In two spatial dimensions, the simplest discrete Laplacian is the 5-point stencil which combines the two 3-point centered schemes in \(x\) and in \(y\). Its accuracy is also \(O(\max\{\Delta x^2, (\Delta y)^2\})\). Designing good absorbing boundary conditions is a somewhat difficult problem that has a long history. The currently most popular solution to this problem is to slightly expand the computational domain using an absorbing, perfectly-matched layer (PML).

More accurate schemes can be obtained from higher-order finite differences. Low-order schemes such as the one explained above typically suffer from unacceptable numerical dispersion at large times. If accuracy is a big concern, spectral methods (spectral elements, Chebyshev polynomials, etc.) are by far the best way to solve wave equations numerically with a controlled, small number of points per wavelength.