Course 18.327 and 1.130
Wavelets and Filter Banks

Mallat pyramid algorithm

Pyramid Algorithm for Computing Wavelet Coefficients

Goal: Given the series expansion for a function $f_j(t)$ in $V_j$

$$f_j(t) = \sum_k a_j[k] \phi_{j,k}(t)$$

how do we find the series

$$f_{j-1}(t) = \sum_k a_{j-1}[k] \phi_{j-1,k}(t)$$
in $V_{j-1}$ and the series

$$g_{j-1}(t) = \sum_k b_{j-1}[k] w_{j-1,k}(t)$$
in $W_{j-1}$ such that

$$f_j(t) = f_{j-1}(t) + g_{j-1}(t)$$
Example: suppose that $\phi(t) = \text{box on } [0,1]$. Then functions in $V_1$ can be written either as a combination of

\[
\phi(2t) = \begin{cases} 
1 & \text{if } 0 \leq t < \frac{1}{2} \\
0 & \text{otherwise}
\end{cases}
\]

or as a combination of

\[
\phi(t) = \begin{cases} 
1 & \text{if } 0 \leq t < 1 \\
0 & \text{otherwise}
\end{cases}
\]

plus a combination of

\[
w(t) = \begin{cases} 
1 & \text{if } 0 \leq t < 1 \\
0 & \text{otherwise}
\end{cases}
\]

Easy to see because

\[
\phi(2t) = \frac{1}{2}[\phi(t) + w(t)]
\]

\[
\phi(2t - 1) = \frac{1}{2}[\phi(t) - w(t)]
\]
Suppose that $f(t)$ is a function in $L^2(R)$. What are the coefficients, $a[j][k]$, of the projection of $f(t)$ on to $V_j$? Call the projection $f_j(t)$,

$$f_j(t) = \sum_k a[j][k] \phi_j[k](t)$$

$a[j][k]$ must minimize the distance between $f(t)$ and $f_j(t)$

$$\int \{f(t) - f_j(t)\}^2 dt = 0$$

$$\int 2 \{f(t) - \sum_l a[j][l] \phi_j[l](t)\} \phi_j[k](t) dt = 0$$

$$a[j][k] = \int f(t) \phi_j[k](t) dt$$

How does $\phi_j[k](t)$ relate to $\phi_{j-1,k}(t)$, $w_{j-1,k}(t)$?

$$\phi(t) = 2 \sum_{\ell=0}^N h_0[\ell] \phi(2t - \ell) \quad \text{refinement equation}$$

$$\phi_{j-1,k}(t) = 2^{(j-1)/2} \phi(2^{j-1}t - k)$$

$$= 2^{(j-1)/2} \cdot 2 \sum_{\ell=0}^N h_0[\ell] \phi(2^{j-1}t - 2k - \ell)$$

$$\phi_{j-1,k}(t) = \sqrt{2} \sum_{\ell=0}^N h_0[\ell] \phi_{j,2k + \ell}(t)$$

Similarly, using the wavelet equation, we have

$$w_{j-1,k}(t) = \sqrt{2} \sum_{\ell=0}^N h_1[\ell] \phi_{j,2k + \ell}(t)$$
Multiresolution decomposition equations

\[ a_{j-1}[n] = \int_{-\infty}^{\infty} f(t) \phi_{j-1,n}(t) \, dt \]

\[ = \sqrt{2} \sum_{\ell} h_0[\ell] \int_{-\infty}^{\infty} f(t) \phi_{j,2n + \ell}(t) \, dt \]

\[ = \sqrt{2} \sum_{\ell} h_0[\ell] a_j[2n + \ell] \]

So

\[ a_{j-1}[n] = \sqrt{2} \sum_{k} h_0[k-2n]a_j[k] \]

→ Convolution with \( h_0[-n] \) followed by downsampling

Similarly

\[ b_{j-1}[n] = \int_{-\infty}^{\infty} f(t) w_{j-1,n}(t) \, dt \]

which leads to

\[ b_{j-1}[n] = \sqrt{2} \sum_{k} h_1[k - 2n] a_j[k] \]
Multiresolution reconstruction equation

Start with
\[ f_j(t) = f_{j-1}(t) + g_{j-1}(t) \]

Multiply by \( \phi_{j,n}(t) \) and integrate
\[
\int_{\infty}^{\infty} f_j(t) \phi_{j,n}(t) \, dt = \int_{\infty}^{\infty} f_{j-1}(t) \phi_{j,n}(t) \, dt + \int_{\infty}^{\infty} g_{j-1}(t) \phi_{j,n}(t) \, dt
\]

So
\[
a_j[n] = \sum_k a_{j-1}[k] \int_{\infty}^{\infty} \phi_{j-1,k}(t) \phi_{j,n}(t) \, dt + \\
\sum_k b_{j-1}[k] \int_{\infty}^{\infty} w_{j-1,k}(t) \phi_{j,n}(t) \, dt
\]

\[
\int_{\infty}^{\infty} \phi_{j-1,k}(t) \phi_{j,n}(t) \, dt = \sqrt{2} \sum_{\ell} h_0[\ell] \int_{\infty}^{\infty} \phi_{j-2k,\ell}(t) \phi_{j,n}(t) \, dt \\
= \sqrt{2} \sum_{\ell} h_0[\ell] \delta[2k + \ell - n] \\
= \sqrt{2} h_0[n - 2k]
\]

Similarly
\[
\int_{\infty}^{\infty} w_{j-1,k}(t) \phi_{j,n}(t) \, dt = \sqrt{2} h_1[n - 2k]
\]

Result:
\[
a_j[n] = \sqrt{2} \sum_k a_{j-1}[k] h_0[n - 2k] + \\
\sqrt{2} \sum_k b_{j-1}[k] h_1[n - 2k]
\]
Filter Bank Representation

Verify that filter bank implements MRA equations:

\[ u_0[n] = \sqrt{2} \sum_k \tilde{h}_0[n-k] a_j[k] \]
\[ b_j[n] = u_1[2n] \]

\[ v_0[n] = \sqrt{2} \sum_{\ell} a_{j-1}[\ell/2] + \sqrt{2} \sum_{\ell} h_1[n - \ell] b_{j-1}[\ell/2] \]

\[ a_{j-1}[n] = u_0[2n] \quad \text{downsample by 2} \]
\[ = \sqrt{2} \sum_k h_0[k - 2n] a_j[k] \]

\[ b_{j-1}[n] = u_1[2n] \]
\[ = \sqrt{2} \sum_k h_1[k - 2n] a_j[k] \]

\[ a_j[n] = \sqrt{2} \sum_{\ell} h_0[n - \ell] v_0[\ell] + \sqrt{2} \sum_{\ell} h_1[n - \ell] v_1[\ell] \]

\[ v_0[\ell] = \begin{cases} a_{j-1}[\ell/2] & ; \ell \text{ even} \\ 0 & ; \text{otherwise} \end{cases} \]