Simple Linear Interpolation

\[ (u_{k_0} + u_{k_2}z^{-2} + u_{k_3}z^{-4}) \times \left( \frac{1}{2}z^2 + 1 + \frac{1}{2}z^{-1} \right) = \]
\[ \frac{1}{2}u_{k_0}z + u_{k_1} + \frac{1}{2}(u_{k_0} + u_{k_2})z^{-1} + u_{k_1}z^{-2} + \frac{1}{2}(u_{k_1} + u_{k_2})z^{-3} + u_{k_2}z^{-4} + \frac{1}{2}u_{k_2}z^{-5} \]

Interpolating Subdivision Schemes

- Given a set of data \( \{ u_{j,k_0}, u_{j,k_1}, \ldots, u_{j,k_N} \} \), find filters \( h_j[k,m] \) such that:
  \[
  \begin{align*}
  u_{j+1,k} &= u_{j,k} \\
  u_{j+1,m} &= \sum_{k \in N(j,m)} h_j[k,m]u_{j,k}
  \end{align*}
  \]
  \( u_{j+1} = Su_j \)

- E.g. two point (linear) scheme
  \[
  u_{j+1,m_i} = \frac{1}{2}(u_{j,k_i} + u_{j,k_{i+1}})
  \]

- Four point (cubic) scheme
  \[
  u_{j+1,m_i} = \frac{1}{16} (-u_{j,k_{i-1}} + 9u_{j,k_i} + 9u_{j,k_{i+1}} - u_{j,k_{i+2}})
  \]

- Generalizes easily to multiple dimensions, non-uniformly spaced points, boundaries, etc.
Interpolating Subdivision Schemes

- Limit curve is an interpolating function

Wavelets From Subdivision

- Limit curves can be used to interpolate data.

On coarse grid

\[ f_j(x) = \sum_{k \in \mathcal{K}(j)} u_{j,k} \varphi_{j,k}(x) \]

On fine grid

\[ f_{j+1}(x) = \sum_{l \in \mathcal{K}(j+1)} u_{j+1,l} \varphi_{j+1,l}(x) \]

Suppose that \( u_{j+1,l} \) is coarsened by subsampling

\[ u_{j,k} = u_{j+1,k} \]

and remaining data is predicted using subdivision

\[ u_{j,m} = u_{j+1,m} - \sum_{k \in N(j,m)} h_j[k,m] u_{j,k} \]
Wavelets From Subdivision

- Does this fit the wavelet framework?

\[
\begin{align*}
    f_{j+1}(x) &= \sum_{l \in \mathcal{K}(j+1)} u_{j+1,l} \varphi_{j+1,l}(x) \quad \text{fine approximation} \\
    &= \sum_{k \in \mathcal{K}(j)} u_{j,k} \varphi_{j,k}(x) + \sum_{m \in \mathcal{M}(j)} u_{j,m} w_{j,m}(x) \\
    &= \underbrace{\sum_{k \in \mathcal{K}(j)} u_{j,k} \varphi_{j,k}(x)}_{\text{coarse approximation}} + \underbrace{\sum_{m \in \mathcal{M}(j)} u_{j,m} w_{j,m}(x)}_{\text{details}}
\end{align*}
\]

If we set \( u_{j,k} = 0 \), \( u_{j,m} = \delta_{m,m'} \), our coarsening/prediction strategy gives

\[
\begin{align*}
    u_{j+1,k} &= u_{j,k} \\
    u_{j+1,m} &= u_{j,m} + \sum_{k \in N(j,m)} h_{j}[k,m] u_{j,k} = 0
\end{align*}
\]

So the “wavelets” are

\[ w_{j,m}(x) = \varphi_{j+1,m}(x) \]

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Wavelets From Subdivision

- Similarly, setting \( u_{j,k} = \delta_{k,k'} \), \( u_{j,m} = 0 \)

\[
\begin{align*}
    u_{j+1,k} &= u_{j,k} \\
    u_{j+1,m} &= \sum_{k \in N(j,m)} h_{j}[k,m] u_{j,k} = h_{j}[k',m]
\end{align*}
\]

produces the refinement equation:

\[
\varphi_{j,k}(x) = \varphi_{j+1,k}(x) + \sum_{m \in n(j,k)} h_{j}[k,m] \varphi_{j+1,m}(x)
\]
Wavelets From Subdivision

- So subdivision schemes naturally lead to hierarchical bases

\[ \begin{align*}
    \psi_k & \quad \Rightarrow \quad \psi_{k_0} \\
    \psi_{k_0} & \quad + \quad \psi_{k_1} \\
    \psi_{k_0} & \quad + \quad \psi_{k_1} \\
    \psi_{k_0} & \quad \Rightarrow \quad \psi_{k_1}
\end{align*} \]

- The coarsening strategy \( u_{j,k} = u_{j+1,k} \) is generally less than ideal – some smoothing (antialiasing) desirable

Accomplished by forcing the wavelet to have one or more vanishing moments

\[ \int w_{j,m}(x)x^kdx = 0, \quad k = 0,1,\ldots,p-1 \]

Larger \( p \) means smaller coefficients \( u_{j,m} \) in wavelet series

\[ f(x) = \sum_{k \in K(j)} u_{j,k} \varphi_{j,k}(x) + \sum_{j=0}^{\infty} \sum_{m \in M(j)} u_{j,m} w_{j,m}(x) \]

\[ u_{j,m} \sim h_j^p f(p)(x_m) \]
Wavelets From Subdivision

• How to improve wavelets using lifting
\[ w_{j,m}^{new}(x) = w_{j,m}(x) - \sum_{k \in \mathbb{K}(j)} s_{j[k,m]} \varphi_{j,k}(x) \]
\[ \varphi_{j,k}(x) \text{ as before} \]

Choose \( s_{j[k,m]} \) to make the moments zero.

• Regardless of the choice for \( s_{j[k,m]} \), \( \varphi_{j,k}(x) \) and \( w_{j,m}^{new}(x) \)
are orthogonal to the dual functions
\[ \tilde{w}_{j,m}^{new}(x) = \tilde{\varphi}_{j+1,m}^{new}(x) - \sum_{k \in \mathbb{N}(j,m)} h_{j[k,m]} \tilde{\varphi}_{j+1,k}^{new}(x) \]
\[ \tilde{\varphi}_{j,k}^{new}(x) = \tilde{\varphi}_{j+1,k}^{new}(x) + \sum_{m \in \mathbb{M}(j)} s_{j[k,m]} \tilde{w}_{j,m}^{new}(x) \]

from which we obtain an improved coarsening strategy:
\[ u_{j,m} = u_{j+1,m} - \sum_{k \in \mathbb{N}(j,m)} h_{j[k,m]} u_{j+1,k} \quad \text{Predict as before} \]
\[ u_{j,k} = u_{j+1,k} + \sum_{m \in \mathbb{M}(j)} s_{j[k,m]} u_{j,m} \quad \text{Then update} \]

Butterfly Subdivision
Loop Subdivision

Finite Elements From Subdivision

- Key difference: subdivision mask is varied so that prediction operation is confined *within* an element

- Limit functions are finite element shape functions
Finite Elements From Subdivision

Scalar subdivision
\[ \frac{1}{16} \{-1, 0, 9, 16, 9, 0, -1\} \]

Finite Element generated from vector subdivision - piecewise polynomial, but lacks smoothness at element boundaries

Smoother vector subdivision schemes also possible

Vector Refinement

- e.g. vector refinement relation for Hermite interpolation functions

\[
\begin{align*}
\{\varphi_{j,k}^\nu(x)\} &= \{\varphi_{j+1,k}^\nu(x)\} + \sum_{m=0}^{n-1} H_j[k,m] \{\varphi_{j+1,m}^\nu(x)\} \\
H_j[k,m] &= \begin{bmatrix} \varphi_k^\nu(x_m) & d\varphi_k^\nu(x_m) \\ \varphi_k^\phi(x_m) & d\varphi_k^\phi(x_m) \end{bmatrix}
\end{align*}
\]

- Wavelets

\[
\sum_{j \in A(j,m)} \int_S \{\varphi_{j,k}^\nu(x) \varphi_{j,k}^\phi(x)\} dx S_j[k,m] = \int_S \{\varphi_{j+1,m}^\nu(x) \varphi_{j+1,m}^\phi(x)\} dx
\]

Cubic subdivision for displacements and rotations