Course 18.327 and 1.130
Wavelets and Filter Banks

Modulation and Polyphase Representations:
Noble Identities;
Block Toeplitz Matrices and Block z-transforms;
Polyphase Examples

Modulation Matrix

Matrix form of PR conditions:

$$[F_0(z)\ \ F_1(z)] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = [\,2z^{-l}\ \ 0\,]$$

1 2 3

Modulation matrix, $H_m(z)$

So

$$[\,F_0(z)\ \ F_1(z)] = [2z^{-l}\ \ 0]\ H_m^{-1}(z)$$

$$H_m^{-1}(z) = \frac{1}{?} \begin{bmatrix} H_1(-z) & -H_0(-z) \\ -H_1(z) & H_0(z) \end{bmatrix}$$

$$? = H_0(z)\,H_1(-z) - H_0(-z)\,H_1(z) \quad \text{(must be non-zero)}$$

$$\Rightarrow F_0(z) = \frac{1}{?} 2z^{-l} \; H_1(-z)$$

$$F_1(z) = -\frac{1}{?} 2z^{-l} \; H_0(-z)$$

Require these to be FIR

Suppose we choose $? = 2z^{-l}$

Then

$$F_0(z) = H_1(-z)$$

$$F_1(z) = -H_0(-z)$$

Synthesis modulation matrix:

Complete the second row of matrix PR conditions by replacing z with –z:

$$\begin{bmatrix} F_0(z) & F_1(z) \\ F_0(-z) & F_1(-z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = 2 \begin{bmatrix} z^{-l} & 0 \\ 0 & (-z)^{-l} \end{bmatrix}$$

1 2 3

Synthesis modulation matrix, $F_m(z)$

Note the transpose convention in $F_m(z)$.

Noble Identities

1. Consider

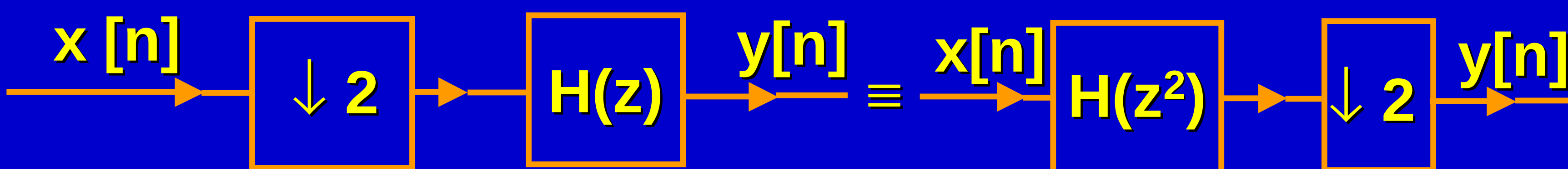
$U(z) = H(z^2)X(z)$

$Y(z) = \frac{1}{2}\{U(z^{\frac{1}{2}}) + U(-z^{\frac{1}{2}})\}$ (downsampling)

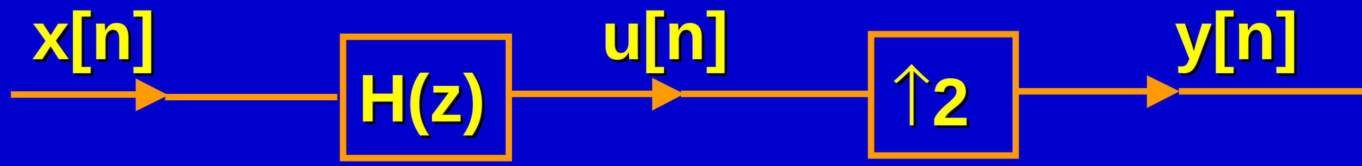
$= \frac{1}{2}\{ H(z)\, X(z^{\frac{1}{2}}) + H(z)\, X\left(-z^{\frac{1}{2}}\right)\}$

$= H(z) \bullet \frac{1}{2}\{X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\}$ $\Rightarrow$ can downsample first

First Noble identity:



2. Consider

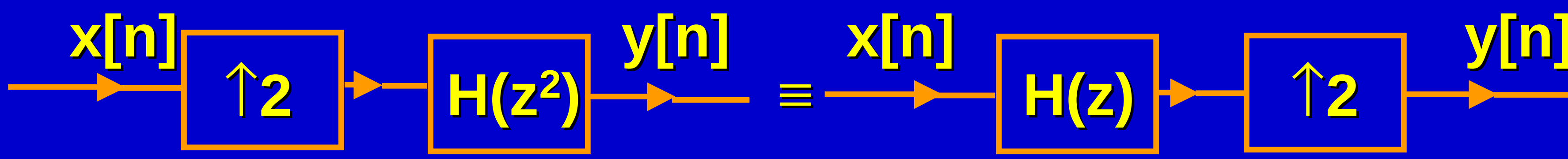


$U(z) = H(z)\,X(z)$

$Y(z) = U(z^2)$ (upsampling)

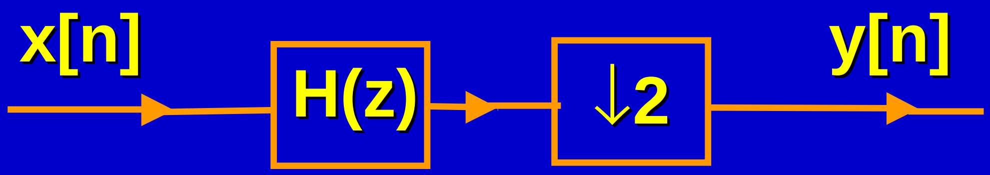
$\quad = H(z^2)\,X(z^2)$ $\Rightarrow$ can upsample first

Second Noble Identity:

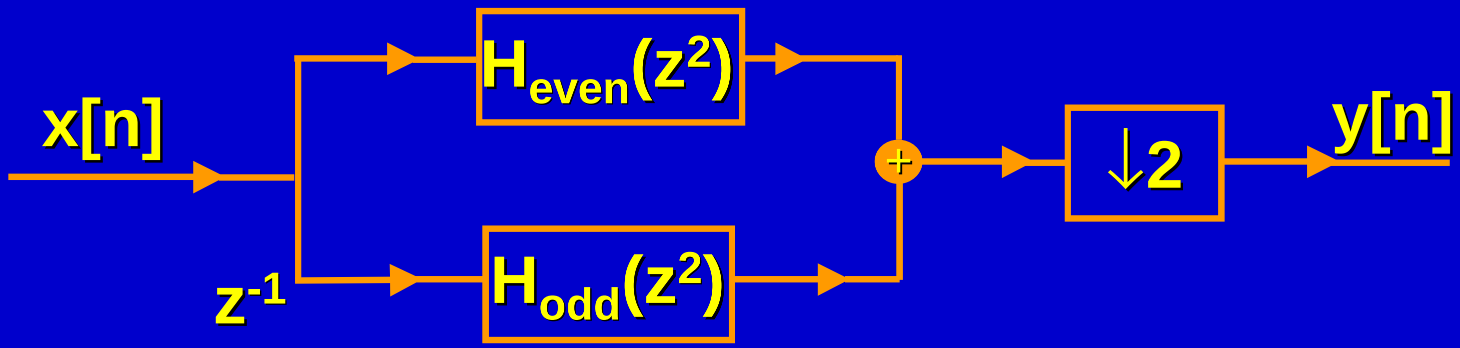


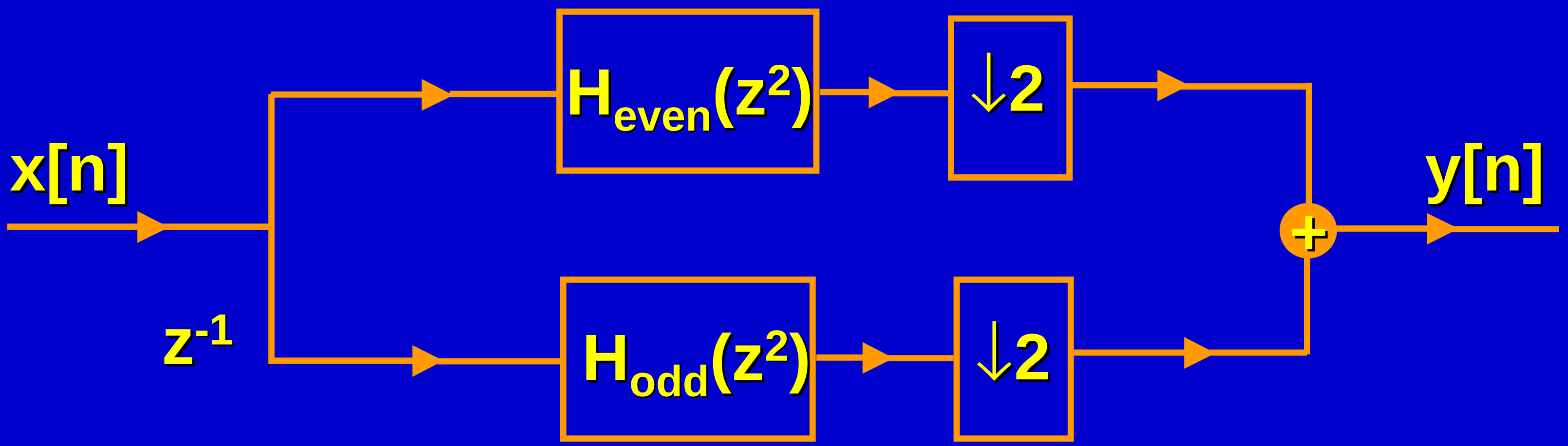
Derivation of Polyphase Form

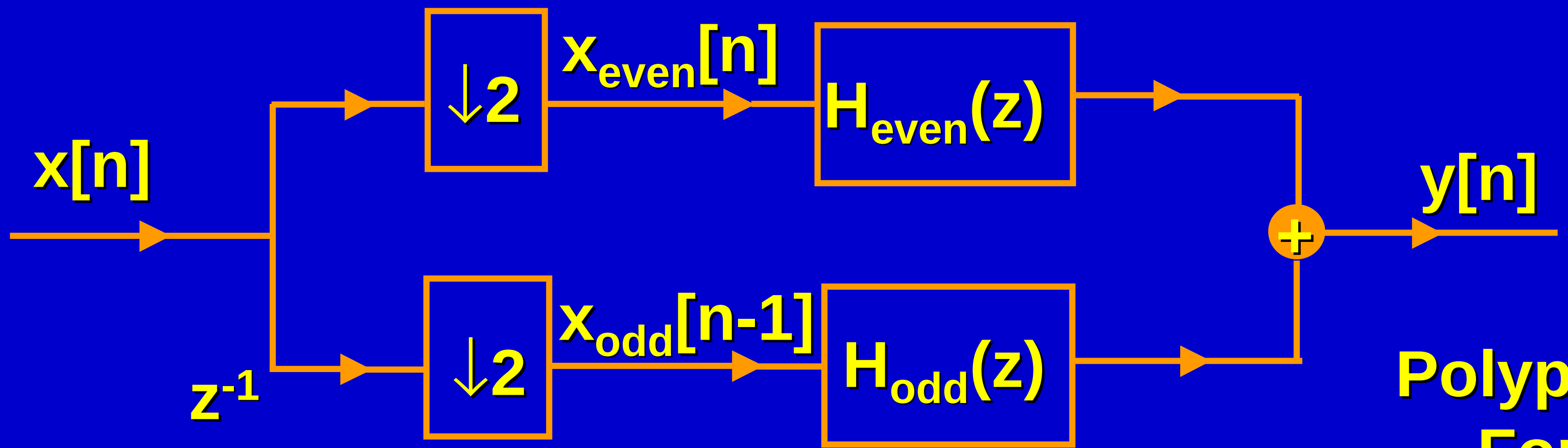
1. Filtering and downsampling:



$$H(z) = H_{even}(z^2) + z^{-1} H_{odd}(z^2); \quad h_{even}[n] = h[2n]$$
$$h_{odd}[n] = h[2n+1]$$

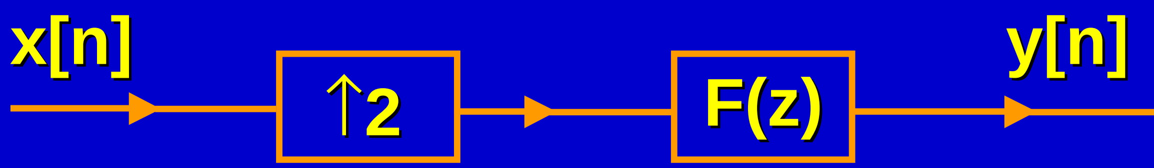




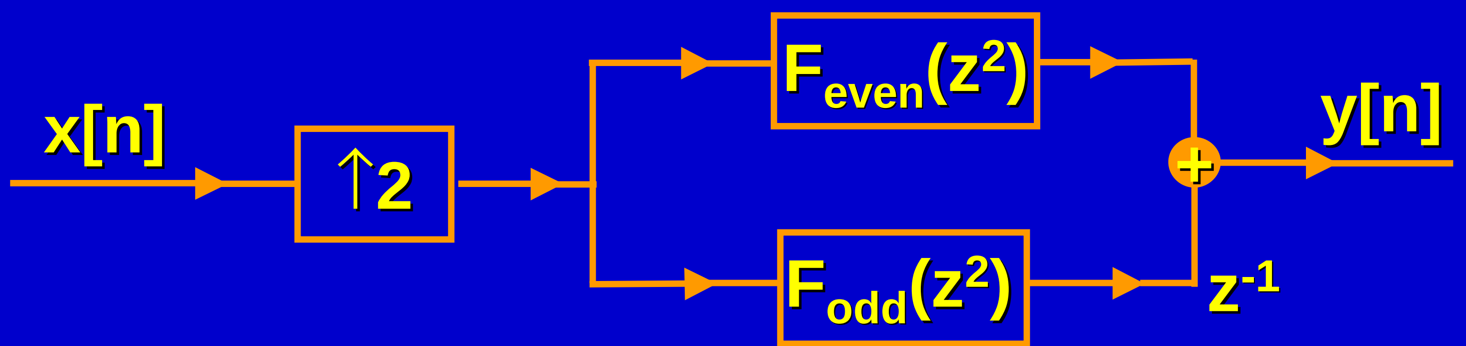


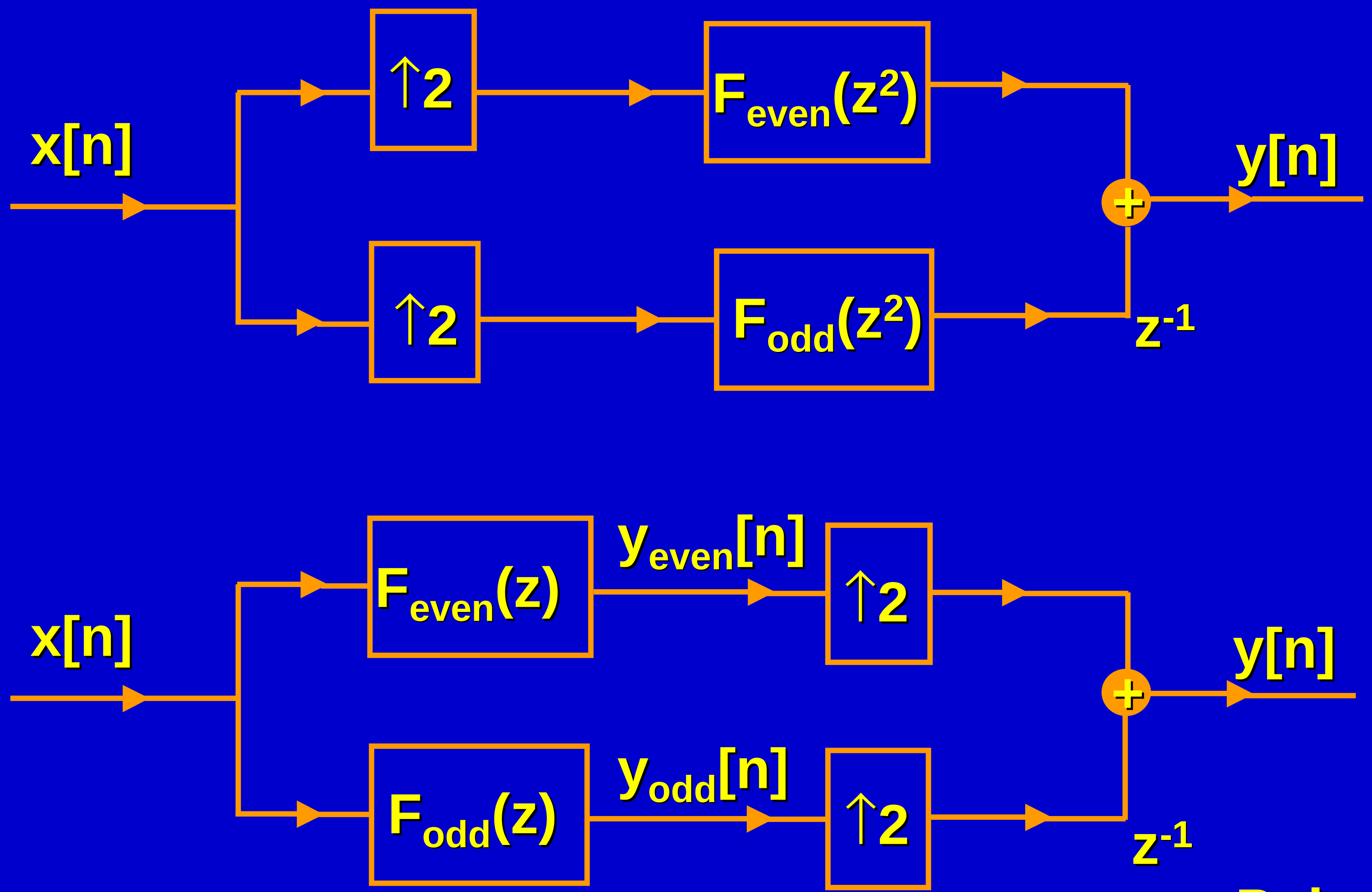
Polyphase Form

2. Upsampling and filtering



$$F(z) = F_{even}(z^2) + z^{-1} F_{odd}(z^2)$$





Polyphase Form

Polyphase Matrix

Consider the matrix corresponding to the analysis filter bank in interleaved form. This is a block Toeplitz matrix:

$$
H_b = \begin{bmatrix}
 & & & \vdots & & & & \\
\cdots & h_0[3] & h_0[2] & h_0[1] & h_0[0] & 0 & 0 & \cdots \\
\cdots & h_1[3] & h_1[2] & h_1[1] & h_1[0] & 0 & 0 & \cdots \\
\cdots & 0 & 0 & h_0[3] & h_0[2] & h_0[1] & h_0[0] & \cdots \\
\cdots & 0 & 0 & h_1[3] & h_1[2] & h_1[1] & h_1[0] & \cdots \\
 & & & \ddots & & & &
\end{bmatrix}
$$

4-tap Example

Taking block z-transform we get:

$$H_p(z) = \begin{bmatrix} h_0[0] & h_0[1] \\ h_1[0] & h_1[1] \end{bmatrix} + z^{-1} \begin{bmatrix} h_0[2] & h_0[3] \\ h_1[2] & h_1[3] \end{bmatrix}$$

$$= \begin{bmatrix} h_0[0] + z^{-1} h_0[2] & h_0[1] + z^{-1} h_0[3] \\ h_1[0] + z^{-1} h_1[2] & h_1[1] + z^{-1} h_1[3] \end{bmatrix}$$

$$= \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix}$$

This is the polyphase matrix for a 2-channel filter bank.

Similarly, for the synthesis filter bank:

$$F_b = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \\ & f_0[0] & f_1[0] & 0 & 0 & \\ & f_0[1] & f_1[1] & 0 & 0 & \\ \cdots & f_0[2] & f_1[2] & f_0[0] & f_1[0] & \cdots \\ & f_0[3] & f_1[3] & f_0[1] & f_1[1] & \\ & 0 & 0 & f_0[2] & f_1[2] & \\ & 0 & 0 & f_0[3] & f_1[3] & \\ & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$F_p(z) = \begin{bmatrix} f_0[0] & f_1[0] \\ f_0[1] & f_1[1] \end{bmatrix} + z^{-1} \begin{bmatrix} f_0[2] & f_1[2] \\ f_0[3] & f_1[3] \end{bmatrix}$$

$$= \begin{bmatrix} F_{0,even}[z] & F_{1,even}[z] \\ F_{0,\ odd}[z] & F_{1,\ odd}[z] \end{bmatrix}$$

Note transpose convention for synthesis polyphase matrix

- **Perfect reconstruction condition in polyphase domain:**

$$F_p(z)\ H_p(z) = I \quad \text{(centered form)}$$

This means that $H_p(z)$ must be invertible for all z on the unit circle, i.e.

$$\det H_p(e^{i\omega}) \neq 0 \text{ for all frequencies } \omega.$$

- **Given that the analysis filters are FIR, the requirement for the synthesis filters to be also FIR is:**

$$\det H_p(z) = z^{-l} \quad \text{(simple delay)}$$

because $H_p^{-1}(z)$ must be a polynomial.

- **Condition for orthogonality: $F_p(z)$ is the transpose of $H_p(z)$, i.e.**

$$H_p^T(z^{-1})\ H_p(z) = I$$

i.e. $H_p(z)$ should be paraunitary.

Relationship between Modulation and Polyphase Matrices

$$H_0(z) = H_{0,even}(z^2) + z^{-1} H_{0,odd}(z^2) \;;\qquad h_{0,even}[n] = h_0[2n],\quad h_{0,odd}[n] = h_0[2n+1]$$

$$H_1(z) = H_{1,even}(z^2) + z^{-1} H_{1,odd}(z^2)$$

Two more equations by replacing z with -z.

So in matrix form:

$$\underbrace{\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}}_{H_m(z)} = \underbrace{\begin{bmatrix} H_{0,even}(z^2) & H_{0,odd}(z^2) \\ H_{1,even}(z^2) & H_{1,odd}(z^2) \end{bmatrix}}_{H_p(z^2)} \begin{bmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{bmatrix}$$

$H_m(z)$ **Modulation matrix** $\quad H_p(z^2)$ **Polyphase matrix**

But

$$\begin{bmatrix} 1 & 1 \\ z^{-1} & -z^{-1} \end{bmatrix} = \begin{bmatrix} 1 & \\ & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

1 2 3 1 2 3

$D_2(z)$ Delay Matrix F_2 2-point DFT Matrix

$$F_N = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & \ldots & w^{N-1} \\ 1 & w^2 & w^4 & \ldots & w^{2(N-1)} \\ . & . & . & & \\ . & . & . & & \\ . & . & . & & \\ 1 & w^{N-1} & w^{2(N-1)} & w^{(N-1)^2} & \end{bmatrix} ; \quad w = e^{i\frac{2\pi}{N}}$$

→ N-point DFT Matrix

$$F_N^{-1} = \frac{1}{N} \overline{F}_N$$

↑ Complex conjugate: replace w with $\overline{w} = e^{-i\frac{2\pi}{N}}$

So, in general

$$H_m(z)\, F_N^{-1} \quad = \quad H_p(z^N)\, D_N(z)$$

N = # of channels in filterbank

(N = 2 in our example)

Polyphase Matrix

Example: Daubechies 4-tap filter

$$h_0[0] = \frac{1+\sqrt{3}}{4\sqrt{2}} \quad h_0[1] = \frac{3+\sqrt{3}}{4\sqrt{2}} \quad h_0[2] = \frac{3-\sqrt{3}}{4\sqrt{2}} \quad h_0[3] = \frac{1-\sqrt{3}}{4\sqrt{2}}$$

$$H_0(z) = \frac{1}{4\sqrt{2}}\{(1+\sqrt{3}) + (3+\sqrt{3})\,z^{-1} + (3-\sqrt{3})\,z^{-2} + (1-\sqrt{3})\,z^{-3}\}$$

$$H_1(z) = \frac{1}{4\sqrt{2}}\{(1-\sqrt{3}) - (3-\sqrt{3})\,z^{-1} + (3+\sqrt{3})z^{-2} - (1+\sqrt{3})z^{-3}\}$$

Time domain:

$$h_0[0]^2 + h_0[1]^2 + h_0[2]^2 + h_0[3]^2 = \frac{1}{32}\{(4 + 2\sqrt{3}) + (12 + 6\sqrt{3}) + (12 - 6\sqrt{3}) + (4 - 2\sqrt{3})\}$$

$$= 1$$

$$h_0[0]\,h_0[2] + h_0[1]\,h_0[3] = \frac{1}{32}\{(2\sqrt{3}) + (-2\sqrt{3})\}$$

$$= 0$$

i.e. filter is orthogonal to its double shifts

Polyphase Domain:

$$H_{0,even}(z) = \frac{1}{4\sqrt{2}} \{(1+\sqrt{3}) + (3-\sqrt{3})\, z^{-1}\}$$

$$H_{0,odd}(z) = \frac{1}{4\sqrt{2}} \{(3+\sqrt{3}) + (1-\sqrt{3})\, z^{-1}\}$$

$$H_{1,even}(z) = \frac{1}{4\sqrt{2}} \{(1-\sqrt{3}) + (3+\sqrt{3})\, z^{-1}\}$$

$$H_{1,odd}(z) = \frac{1}{4\sqrt{2}} \{-(3-\sqrt{3}) - (1+\sqrt{3})\, z^{-1}\}$$

$$H_p(z) = \frac{1}{4\sqrt{2}} \underbrace{\begin{bmatrix} 1+\sqrt{3} & 3+\sqrt{3} \\ 1-\sqrt{3} & -(3-\sqrt{3}) \end{bmatrix}}_{A} + \frac{1}{4\sqrt{2}} \underbrace{\begin{bmatrix} 3-\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(1+\sqrt{3}) \end{bmatrix}}_{B} z^{-1}$$

1 2 3

1 2 3

A

B

$$H_p(z) = A + B z^{-1}$$

$$H_p^T(z^{-1})\, H_p(z) = (A^T + B^T z)(A + Bz^{-1})$$
$$= (A^TA + B^TB) + A^TBz^{-1} + B^TAz$$

$$A^TA = \frac{1}{4\sqrt{2}} \begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(3-\sqrt{3}) \end{bmatrix} \frac{1}{4\sqrt{2}} \begin{bmatrix} 1+\sqrt{3} & 3+\sqrt{3} \\ 1-\sqrt{3} & -(3-\sqrt{3}) \end{bmatrix}$$

$$= \frac{1}{32} \begin{bmatrix} (4+2\sqrt{3}) + (4-2\sqrt{3}) & (6+4\sqrt{3}) - (6-4\sqrt{3}) \\ (6+4\sqrt{3}) - (6-4\sqrt{3}) & (12+6\sqrt{3}) + (12-6\sqrt{3}) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \sqrt{3}/4 \\ \sqrt{3}/4 & \frac{3}{4} \end{bmatrix}$$

$$B^TB = \frac{1}{4\sqrt{2}}\begin{bmatrix} 3-\sqrt{3} & 3+\sqrt{3} \\ 1-\sqrt{3} & -(1+\sqrt{3}) \end{bmatrix} \frac{1}{4\sqrt{2}}\begin{bmatrix} 3-\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(1+\sqrt{3}) \end{bmatrix}$$

$$= \frac{1}{32}\begin{bmatrix} (12-6\sqrt{3})+(12+6\sqrt{3}) & (6-4\sqrt{3})-(6+4\sqrt{3}) \\ (6-4\sqrt{3})-(6+4\sqrt{3}) & (4-2\sqrt{3})+(4+2\sqrt{3}) \end{bmatrix}$$

$$= \begin{bmatrix} 3/4 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & 1/4 \end{bmatrix}$$

$$\Rightarrow A^TA + B^TB = I$$

$$A^T B = \frac{1}{4\sqrt{2}} \begin{bmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(3-\sqrt{3}) \end{bmatrix} \frac{1}{4\sqrt{2}} \begin{bmatrix} 3-\sqrt{3} & 1-\sqrt{3} \\ 3+\sqrt{3} & -(1+\sqrt{3}) \end{bmatrix}$$

$$= \frac{1}{32} \begin{bmatrix} (2\sqrt{3}) + (-2\sqrt{3}) & (-2) - (-2) \\ (6) - (6) & (-2\sqrt{3}) + (2\sqrt{3}) \end{bmatrix}$$

$$= \quad 0$$

$$B^T A \quad = \quad (A^T B)^T \quad = \; 0$$

So

$H_p^T(z^{-1}) \; H_p(z) \; = \; I$ i.e. $H_p(z)$ is a Paraunitary Matrix

Modulation domain:

$$H_0(z)\ H_0(z^{-1}) = P(z) = \frac{1}{16}\ (-z^3 + 9z + 16 + 9z^{-1} - z^{-3})$$

$$H_0(-z)\ H_0(-z^{-1}) = P(-z) = \frac{1}{16}\ (z^3 - 9z + 16 - 9z^{-1} + z^{-3})$$

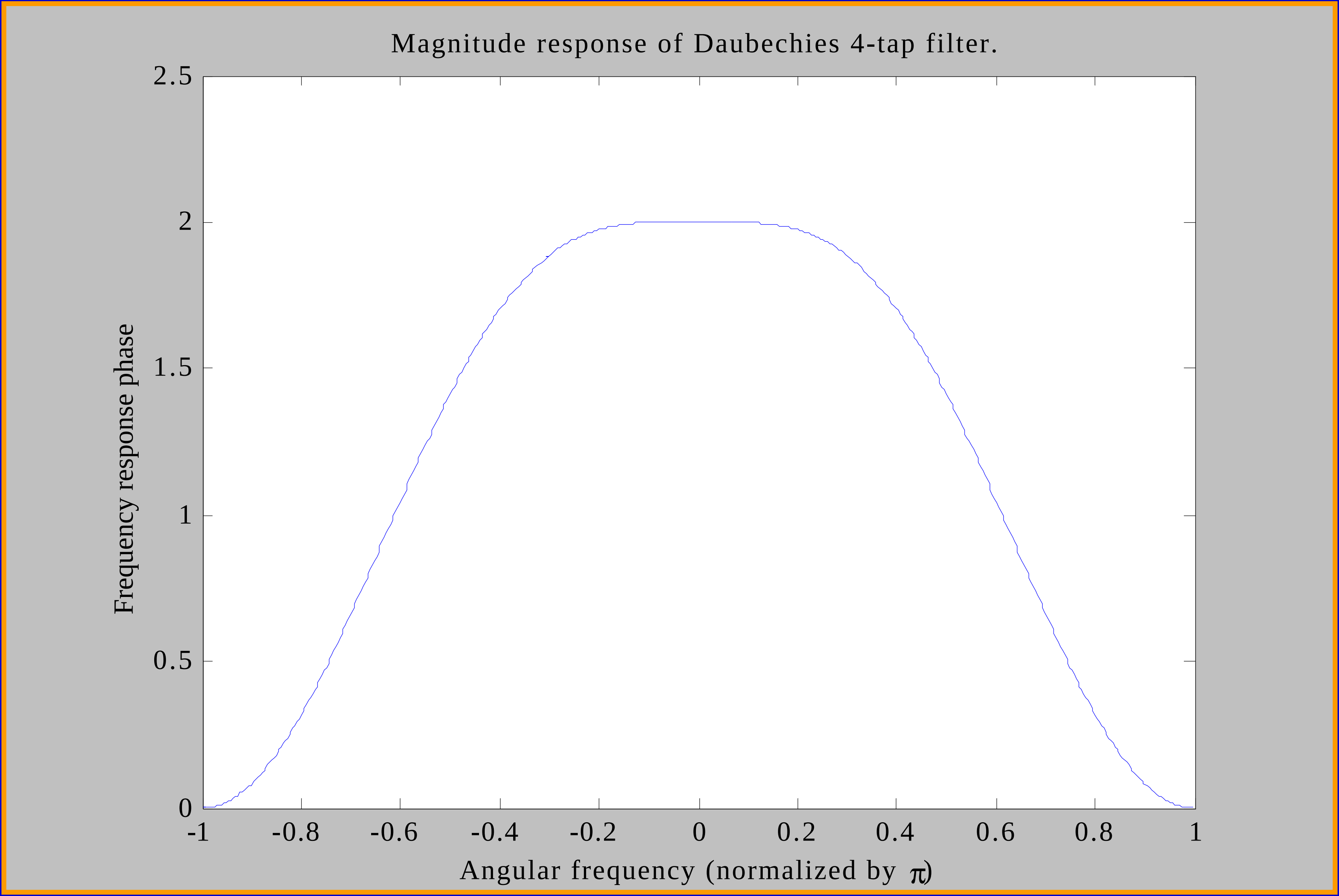
So

$$H_0(z)\ H_0(z^{-1}) + H_0(-z)\ H_0(-z^{-1}) = 2$$

i.e.

$$|H_0(\omega)|\hat{}2 + |H_0(\omega + \pi)|\hat{}2 = 2$$

Magnitude Response of Daubechies 4-tap filter.



Phase response of Daubechies 4-tap filter.