Course 18.327 and 1.130
Wavelets and Filter Banks

Maxflat Filters: Daubechies and Meyer Formulas.
Spectral Factorization
Formulas for the Product Filter

Halfband condition:
\[ P(\omega) + P(\omega + \pi) = 2 \]
Also want \( P(\omega) \) to be lowpass and \( p[n] \) to be symmetric.

Daubechies’ Approach

Design a polynomial, \( \tilde{P}(y) \), of degree \( 2p - 1 \), such that
\[ P(0) = 2 \]
\[ \tilde{P}^{(l)}(0) = 0; \ l = 1, 2, \ldots, p - 1 \]
\[ \tilde{P}^{(l)}(1) = 0; \ l = 0, 1, \ldots, p - 1 \]
Can achieve required flatness at $y = 1$ by including a term of the form $(1 - y)^p$ i.e.

$$\tilde{P}(y) = 2(1 - y)^p B_p(y)$$

Where $B_p(y)$ is a polynomial of degree $p - 1$.

How to choose $B_p(y)$?

Let $B_p(y)$ be the binomial series expansion for $(1 - y)^{-p}$, truncated after $p$ terms:

$$B_p(y) = 1 + py + \frac{p(p + 1)}{2} y^2 + \ldots + \binom{2p - 2}{p - 1} y^{p-1}$$

$$= (1 - y)^{-p} + O(y^p)$$

$<\text{Higher order terms}$
\[(1 - y)^{-1} = \sum_{k=0}^{\infty} y^k\]
\[(1 - y)^{-p} = \sum_{k=0}^{\infty} \binom{p + k - 1}{k} y^k\]

\[|y| < 1\]

Then

\[\tilde{P}(y) = 2(1 - y)^p[(1-y)^{-p} + O(y^p)]\]
\[= 2 + O(y^p)\]
Thus
\[ P^{(l)}(0) = 0 ; \quad l = 1, 2, \ldots, p-1 \]
So we have
\[ \tilde{P}(y) = 2 (1-y)^p \sum_{k=0}^{p-1} \left( \begin{array}{c} p + k - 1 \\ k \end{array} \right) y^k \]
Now let
\[ y = \left( \frac{1 - e^{i\omega}}{2} \right) \left( \frac{1 - e^{-i\omega}}{2} \right) \text{ maintains symmetry} \]
\[ y = \frac{1 - \cos \omega}{2} \]
Thus
\[ P(\omega) = \tilde{P} \left( \frac{1 - \cos \omega}{2} \right) \]
\[ = 2 \left( \frac{1 + \cos \omega}{2} \right)^p \sum_{k=0}^{p-1} \left( \begin{array}{c} p + k + 1 \\ k \end{array} \right) \left( \frac{1 - \cos \omega}{2} \right)^k \]
z domain:

\[ P(z) = 2 \left( \frac{1 + z}{2} \right)^p \left( \frac{1 + z^{-1}}{2} \right)^p \sum_{k = 0}^{p-1} \binom{p + k - 1}{k} \left( \frac{1 - z}{2} \right)^k \left( \frac{1 - z^{-1}}{2} \right)^k \]
Meyer’s Approach

Work with derivative of $\tilde{P}(y)$:

$$\tilde{P}'(y) = - C' y^{p-1} (1 - y)^{p-1}$$

So

$$\tilde{P}(y) = 2 - C' \int_0^y y^{p-1} (1-y)^{p-1} \, dy \quad (\tilde{P}(0) = 2)$$

Then

$$P(\omega) = 2 - C' \int_0^\omega \left( \frac{1 - \cos \omega}{2} \right)^{p-1} \left( \frac{1 + \cos \omega}{2} \right)^{p-1} \frac{\sin \omega}{2} \, d\omega$$

$$= 2 - C' \int_0^\omega \left( \frac{1 - \cos^2 \omega}{2} \right)^{p-1} \frac{\sin \omega}{2} \, d\omega$$

i.e.

$$P(\omega) = 2 - C \int_0^\omega \sin^{2p-1} \omega \, d\omega$$
Spectral Factorization

Recall the halfband condition for orthogonal filters:

**z domain:**
\[ H_0(z) H_0(z^{-1}) + H_0(-z) H_0(-z^{-1}) = 2 \]

**Frequency domain:**
\[ |H_0(\omega)|^2 + |H_0(\omega + \pi)|^2 = 2 \]

The product filter for the orthogonal case is
\[ P(z) = H_0(z) H_0(z^{-1}) \]
\[ P(\omega) = |H_0(\omega)|^2 \quad \Rightarrow \quad P(\omega) \geq 0 \]
\[ p[n] = h_0[n] * h_0[-n] \quad \Rightarrow \quad p[n] = p[-n] \]

The spectral factorization problem is the problem of finding \( H_0(z) \) once \( P(z) \) is known.
Consider the distribution of the zeros (roots) of $P(z)$.

- **Symmetry of $p[n]$** $\Rightarrow$ $P(z) = P(z^{-1})$
  
  If $z_0$ is a root then so is $z_0^{-1}$.

- **If $p[n]$ are real, then the roots appear in complex, conjugate pairs.**

\[
(1 - z_0 z^{-1})(1 - z_0^* z^{-1}) = 1 - (z_0 + z_0^*) z^{-1} + (z_0 z_0^*) z^{-2}
\]

real real
If the zero $z_0$ is grouped into the spectral factor $H_0(z)$, then the zero $1/z_0$ must be grouped into $H_0(z^{-1})$.

$\Rightarrow h_0[n]$ cannot be symmetric.
Daubechies’ choice: Choose $H_0(z)$ such that
(i) all its zeros are inside or on the unit circle.
(ii) it is causal.
i.e. $H_0(z)$ is a minimum phase filter.

Example:

$P(z) = H_0(z) = H_0(z^{-1})$

$P(z)$

$H_0(z)$ (Minimum phase)

$H_0(z^{-1})$ (Maximum phase)
Practical Algorithms:

1. **Direct Method**: compute the roots of $P(z)$ numerically.

2. **Cepstral Method**:
   
   First factor out the zeros which lie on the unit circle

   
   $$P(z) = [(1 + z^{-1})(1 + z)]^p Q(z)$$

   Now we need to factor $Q(z)$ into $R(z) R(z^{-1})$ such that

   i. $R(z)$ has all its zeros inside the unit circle.

   ii. $R(z)$ is causal.
Then use logarithms to change multiplication into addition:

\[ Q(z) = R(z) \cdot R(z^{-1}) \]

\[ \ln Q(z) = \ln R(z) + \ln R(z^{-1}) \]

\[ \hat{Q}(z) = \hat{R}(z) + \hat{R}(z^{-1}) \]

Take inverse z transforms:

\[ \hat{q}[n] = \hat{r}[n] + \hat{r}[-n] \]

Complex cepstrum of \( q[n] \)
Example:

$R(z)$ has all its zeros and all its poles inside the unit circle, so $\hat{R}(z)$ has all its singularities inside the unit circle. ($\ln 0 = -\infty$, $\ln \infty = \infty$.)
All singularities inside the unit circle leads to a causal sequence, e.g.

\[ X(z) = \frac{1}{1 - z_k z^{-1}} \]

Pole at \( z = z_k \)

\[ X(\omega) = \frac{1}{1 - z_k -i\omega} \]

If \( |z_k| < 1 \), we can write

\[ X(\omega) = \sum_{n=0}^{\infty} (z_k)^n e^{-i\omega n} \]

\( \Rightarrow x[n] \) is causal

So \( \hat{r}[n] \) is the causal part of \( \hat{q}[n] \):

\( \hat{r}[n] = \begin{cases} \frac{1}{2} \hat{q}[0] &; n = 0 \\ \hat{q}[n] &; n > 0 \\ 0 &; n < 0 \end{cases} \)
Algorithm:
Given the coefficients q[n] of the polynomial Q(z):
i. Compute the M-point DFT of q[n] for a sufficiently large M.

\[ Q[k] = \sum_n q[n] e^{-i \frac{2\pi kn}{M}} ; \quad 0 \leq k < M \]

ii. Take the logarithm.

\[ \hat{Q}[k] = \ln (Q[k]) \]

iii. Determine the complex cepstrum of q[n] by computing the IDFT.

\[ \hat{q}[n] = \frac{1}{M} \sum_{k=0}^{M-1} \hat{Q}[k] e^{i \frac{2\pi nk}{M}} \]
iv. Find the causal part of $\hat{q}[n]$.

$$
\hat{r}[n] = \begin{cases} 
\frac{1}{2} \hat{q}[0] & ; \quad n = 0 \\
\hat{q}[n] & ; \quad n > 0 \\
0 & ; \quad n < 0
\end{cases}
$$

v. Determine the DFT of $r[n]$ by computing the exponent of the DFT of $\hat{r}[n]$.

$$
R[k] = \exp (\hat{R}[k]) = \exp \left( \sum_{k=0}^{M-1} \hat{r}[n] e^{-j \frac{2\pi}{M} k n} \right) ; 0 \leq k < M
$$
vi. Determine the DFT of $h_0[n]$, by including half the zeros at $z = -1$.

$$H_0[k] = R[k] \left( 1 + e^{-i \frac{2\pi k}{M}} \right)^p$$

vii. Compute the IDFT to get $h_0[n]$.

$$h_0[n] = \frac{1}{M} \sum_{k=0}^{M-1} H_0[k] e^{i \frac{2\pi n}{M} k}$$