18.335 Practice Midterm

1. (5 points) Let $A$ be real symmetric and positive semidefinite, i.e. $x^T A x \geq 0$ for all $x \neq 0$. Show that if the diagonal of $A$ is zero, then $A$ is zero.

2. (5 points) Show that if

$$ Y = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} $$

then $\kappa_F(Y) = 2n + \|Z\|^2_F$.

3. Let

$$ T = \begin{bmatrix} a_1 & b_1 \\ c_1 & \ddots & \ddots \\ & \ddots & \ddots & b_{n-1} \\ c_{n-1} & a_n \end{bmatrix} $$

be a real, $n$-by-$n$, nonsymmetric tridiagonal matrix where $c_i b_i > 0$ for all $1 \leq i \leq n - 1$. Show that the eigenvalues of $T$ are real (5 points) and distinct (5 points).

Hint: Find a diagonal matrix $D$ such that $C = DTD^{-1}$ is symmetric. Then argue about the rank of $C - \lambda I$.

4. (5 points) Let $A$ be symmetric positive definite matrix with Cholesky factor $C$, i.e. $A = C^T C$. Show that $\|A\|_2 = \|C\|_2^2$.

5. (5 points) If $A$ and $B$ are real symmetric positive definite matrices then decide whether the following are true, justifying your results:

- $A + B$ is symmetric positive definite.
- $A \cdot B$ is symmetric positive definite.

6. (5 points) Prove that $\det(I + xy^T) = 1 + x^T y$. 
