Well-Posedness

Def.: A PDE is called well-posed (in the sense of Hadamard), if

1. a solution exists
2. the solution is unique
3. the solution depends continuously on the data
   (initial conditions, boundary conditions, right hand side)

Careful: Existence and uniqueness involves boundary conditions

Ex.: $u_{xx} + u = 0$
   a) $u(0) = 0, u(\frac{\pi}{2}) = 1 \Rightarrow$ unique solution $u(x) = \sin(x)$
   b) $u(0) = 0, u(\pi) = 1 \Rightarrow$ no solution
   c) $u(0) = 0, u(\pi) = 0 \Rightarrow$ infinitely many solutions: $u(x) = A \sin(x)$

Continuous dependence depends on considered metric/norm.
We typically consider $|| \cdot ||_{L^\infty}, || \cdot ||_{L^2}, || \cdot ||_{L^1}$.

Ex.: \[
\begin{cases}
  u_t = u_{xx} & \text{heat equation} \\
  u(0, t) = u(1, t) = 0 & \text{boundary conditions} \\
  u(x, 0) = u_0(x) & \text{initial conditions}
\end{cases}
\]
   well-posed

\[
\begin{cases}
  u_t = -u_{xx} & \text{backwards heat equation} \\
  u(0, t) = u(1, t) & \text{boundary conditions} \\
  u(x, 0) = u_0(x) & \text{initial conditions}
\end{cases}
\]
   no continuous dependence
   on initial data [later]

Notions of Solutions

Classical solution
$k^{th}$ order PDE $\Rightarrow u \in C^k$

Ex.: $\nabla^2 u = 0 \Rightarrow u \in C^\infty$

\[
\begin{cases}
  u_t + u_x = 0 \\
  u(x, 0) \in C^1
\end{cases}
\]
   $\Rightarrow u(x, t) \in C^1$

Weak solution
$k^{th}$ order PDE, but $u \notin C^k$. 
Ex.: Discontinuous coefficients
\[
\begin{align*}
\begin{cases}
  (b(x)u_x)_x = 0 \\
  u(0) = 0 \\
  u(1) = 1 \\
  b(x) = \begin{cases} 
    1 & x < \frac{1}{2} \\
    2 & x \geq \frac{1}{2} 
  \end{cases}
\end{cases}
\end{align*}
\Rightarrow u(x) = \begin{cases} 
  \frac{4}{3}x - \frac{1}{3} & x < \frac{1}{2} \\
  \frac{1}{3} & x \geq \frac{1}{2}
\end{cases}
\]

Ex.: Conservation laws
\[
u_t + \left( \frac{1}{2} u^2 \right)_x = 0 \quad \text{Burgers’ equation}
\]

![Image of Burgers' equation solution with shocks](Image by MIT OpenCourseWare.)

**Fourier Methods for Linear IVP**

IVP = initial value problem
\[
\begin{align*}
  u_t &= u_x \quad \text{advection equation} \\
  u_t &= u_{xx} \quad \text{heat equation} \\
  u_t &= u_{xxx} \quad \text{Airy’s equation} \\
  u_t &= u_{xxxx}
\end{align*}
\]

a) on whole real axis: \( u(x, t) = \int_{-\infty}^{+\infty} e^{ixw} \hat{u}(w, t) dw \quad \text{Fourier transform} \)

b) periodic case \( x \in [-\pi, \pi] : u(x, t) = \sum_{k=-\infty}^{+\infty} \hat{u}_k(t)e^{ikx} \theta \text{ Fourier series (FS)} \)

Here case b).

PDE: \( \frac{\partial u}{\partial t}(x, t) - \frac{\partial^n u}{\partial x^n}(x, t) = 0 \)

insert FS: \( \sum_{k=-\infty}^{+\infty} \left( \frac{d\hat{u}_k}{dt}(t) - (ik)^n \hat{u}_k(t) \right) e^{ikx} = 0 \)

Since \( (e^{ikx})_{k \in \mathbb{Z}} \) linearly independent:
\[
\frac{d\hat{u}_k}{dt} = (ik)^n \hat{u}_k(t) \quad \text{ODE for each Fourier coefficient}
\]

2
Solution:  \[ \hat{u}_k(t) = e^{ik^2t} \hat{u}_k(0) \]

Fourier coefficient of initial conditions: \( \hat{u}_k(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u_0(x)e^{-ikx}dx \)

\[ \Rightarrow u(x,t) = \sum_{k=-\infty}^{+\infty} \hat{u}_k(0)e^{ikx}e^{ik^2t} \]

\( n = 1: \)  \( u(x,t) = \sum_k \hat{u}_k(0)e^{ik(x+t)} \) all waves travel to left with velocity 1

\( n = 2: \)  \( u(x,t) = \sum_k \hat{u}_k(0)e^{ikx}e^{-k^2t} \) frequency \( k \) decays with \( e^{-k^2t} \)

\( n = 3: \)  \( u(x,t) = \sum_k \hat{u}_k(0)e^{ik(x-k^2t)} \) frequency \( k \) travels to right with velocity \( k^2 \rightarrow \) dispersion

\( n = 4: \)  \( u(x,t) = \sum_k \hat{u}_k(0)e^{ikx}e^{k^4t} \) all frequencies are amplified \( \rightarrow \) unstable

Message:
For linear PDE IVP, study behavior of waves \( e^{ikx} \).

The ansatz \( u(x,t) = e^{-iwt}e^{ikx} \) yields a dispersion relation of \( w \) to \( k \).

The wave \( e^{ikx} \) is transformed by the growth factor \( e^{-ik^2t} \).

\textbf{Ex.}:

wave equation: \( u_{tt} = c^2u_{xx} \)  \( w = \pm ck \) conservative \( |e^{\pm ikt}| = 1 \)

heat equation: \( u_t = du_{xx} \)  \( w = -idk^2 \) dissipative \( |e^{-dk^2t}| \rightarrow 0 \)

conv.-diffusion: \( u_t = cu_x + du_{xx} \)  \( w = -ck-idk^2 \) dissipative \( |e^{iwt}e^{-dk^2t}| \rightarrow 0 \)

Schrödinger: \( iu_t = u_{xx} \)  \( w = -k^2 \) dispersive \( |e^{ik^2t}| = 1 \)

Airy equation: \( u_t = u_{xxx} \)  \( w = k^3 \) dispersive \( |e^{-ik^3t}| = 1 \)