14. Instability of Superposed Fluids

Figure 14.1: Wind over water: A layer of fluid of density $\rho^+$ moving with relative velocity $V$ over a layer of fluid of density $\rho^-$. Define interface: $h(x, y, z) = z - \eta(x, y) = 0$ so that $\nabla h = (-\eta_x, -\eta_y, 1)$.

The unit normal is given by
\[
\hat{n} = \frac{\nabla h}{|\nabla h|} = \frac{(-\eta_x, -\eta_y, 1)}{(\eta_x^2 + \eta_y^2 + 1)^{1/2}}
\]

(14.1)

Describe the fluid as inviscid and irrotational, as is generally appropriate at high $Re$.

Basic state: $\eta = 0$, $u = \nabla \phi$, $\phi = \pm \frac{1}{2} V z$ for $z \pm$.

Perturbed state: $\phi = \mp \frac{1}{2} V + \phi_\pm$ in $z \pm$, where $\phi_\pm$ is the perturbation field.

Solve
\[
\nabla \cdot u = \nabla^2 \phi_\pm = 0
\]

subject to BCs:

1. $\phi_\pm \to 0$ as $z \to \pm \infty$

2. Kinematic BC: $\frac{\partial \eta}{\partial t} = u \cdot \hat{n}$,

   where
\[
u = \nabla \left( \mp \frac{1}{2} V + \phi_\pm \right) = \mp \frac{1}{2} V \hat{x} + \frac{\partial \phi_\pm}{\partial x} \hat{x} + \frac{\partial \phi_\pm}{\partial y} \hat{y} + \frac{\partial \phi_\pm}{\partial z} \hat{z}
\]

(14.3)

from which
\[
\frac{\partial \eta}{\partial t} = \left( \mp \frac{1}{2} V + \frac{\partial \phi_\pm}{\partial x} \right) (-\eta_x) + \frac{\partial \phi_\pm}{\partial y} (-\eta_y) + \frac{\partial \phi_\pm}{\partial z}
\]

(14.4)

Linearize: assume perturbation fields $\eta$, $\phi_\pm$ and their derivatives are small and therefore can neglect their products.

Thus $\eta \approx (-\eta_x, -\eta_y, 1)$ and $\frac{\partial \eta}{\partial t} = \pm \frac{1}{2} V \eta_x + \frac{\partial \phi_\pm}{\partial x} \Rightarrow$
\[
\frac{\partial \phi_\pm}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{1}{2} V \frac{\partial \eta}{\partial x} \text{ on } z = 0
\]

(14.5)

3. Normal Stress Balance: $p_- - p_+ = \sigma \nabla \cdot \hat{n}$ on $z = \eta$.

   Linearize: $p_- - p_+ = -\sigma (\eta_{xx} + \eta_{yy})$ on $z = 0$. 55
We now deduce $p_{\pm}$ from time-dependent Bernoulli:

$$\rho \frac{\partial \phi_{\pm}}{\partial t} + \frac{1}{2} \rho u^2 + p + \rho g z = f(t) \tag{14.6}$$

where $u^2 = \frac{1}{4} V^2 \mp V \frac{\partial \phi_{\pm}}{\partial x} + H.O.T.$

Linearize:

$$\rho \frac{\partial \phi_{\pm}}{\partial t} + \frac{1}{2} \rho u \left( \mp V \frac{\partial \phi_{\pm}}{\partial x} \right) + p_{\pm} + \rho \pm g \eta = G(t) \tag{14.7}$$

so

$$p_- - p_+ = (\rho_+ - \rho_-) \eta + \rho_+ \frac{\partial \phi_+}{\partial t} - \rho_- \frac{\partial \phi_-}{\partial t} + \frac{V}{2} (\rho_+ \frac{\partial \phi_+}{\partial x} - \rho_- \frac{\partial \phi_-}{\partial x}) = -\sigma (\eta_{xx} + \eta_{yy}) \tag{14.8}$$

is the linearized normal stress BC. Seek normal mode (wave) solutions of the form

$$\eta = \eta_0 e^{i \alpha x + i \beta y + \omega t} \tag{14.9}$$

$$\phi_{\pm} = \phi_{0\pm} e^{\mp k z} e^{i \alpha x + i \beta y + \omega t} \tag{14.10}$$

where $\nabla^2 \phi_{\pm} = 0$ requires $k^2 = \alpha^2 + \beta^2$.

Apply kinematic BC: $\frac{\partial \phi_{\pm}}{\partial z} = 0 \Rightarrow \mp k \phi_{0\pm} = \omega \eta_0 + \frac{1}{2} i \alpha V \eta_0 \tag{14.11}$

Normal stress BC:

$$k^2 \sigma \eta_0 = -g (\rho_- - \rho_+) \eta_0 + \omega (\rho_+ \phi_{0+} - \rho_- \phi_{0-}) + \frac{1}{2} i \alpha V (\rho_+ \phi_{0+} + \rho_- \phi_{0-}) \tag{14.12}$$

Substitute for $\phi_{0\pm}$ from (14.11):

$$-k^4 \sigma = \omega \left[ \rho_+ (\omega - \frac{1}{2} i \alpha V) + \rho_- (\omega + \frac{1}{2} i \alpha V) \right] + gk (\rho_- - \rho_+) + \frac{1}{2} i \alpha V \left[ \rho_+ (\omega - \frac{1}{2} i \alpha V) + \rho_- (\omega + \frac{1}{2} i \alpha V) \right]$$

so

$$\omega^2 + i \alpha V \left( \frac{\rho_- - \rho_+}{\rho_- + \rho_+} \right) \omega - \frac{1}{4} \alpha^2 V^2 + k^2 C_0^2 = 0 \tag{14.13}$$

where $C_0^2 \equiv \frac{\rho_- - \rho_+}{\rho_- + \rho_+} \frac{g}{k} + \frac{\sigma}{\rho_- + \rho_+} \frac{k}{k}$.

**Dispersion relation:** we now have the relation between $\omega$ and $k$

$$\omega = \frac{1}{2} \sqrt{ \left( \frac{\rho_- - \rho_+}{\rho_- + \rho_+} \right) k \cdot V + \left[ \frac{\rho_- - \rho_+}{\rho_- + \rho_+} \right]^2 (k \cdot V)^2 - k^2 C_0^2 }^{1/2} \tag{14.14}$$

where $k = (\alpha, \beta)$, $k^2 = \alpha^2 + \beta^2$.

The system is **UNSTABLE** if $\text{Re} (\omega) > 0$, i.e. if

$$\frac{\rho_- - \rho_+}{\rho_- + \rho_+} (k \cdot V)^2 > k^2 C_0^2 \tag{14.15}$$

**Squires Theorem:**
Disturbances with wave vector $k = (\alpha, \beta)$ parallel to $V$ are most unstable. This is a general property of shear flows.

We proceed by considering two important special cases, Rayleigh-Taylor and Kelvin-Helmholtz instability.
14.1 Rayleigh-Taylor Instability

We consider an initially static system in which heavy fluid overlies light fluid: \( \rho_+ > \rho_- \), \( V = 0 \). Via (14.15), the system is unstable if

\[
C_0^2 = \frac{\rho_- - \rho_+}{\rho_+ \rho_-} \frac{g}{k} + \frac{\sigma}{\rho_+ + \rho_-} k < 0
\]  

(14.16)

i.e. if \( \rho_+ - \rho_- > \frac{\sigma k^2}{g} = \frac{4\pi^2 \sigma}{9 \pi^2} \).

Thus, for instability, we require: \( \lambda > 2\pi \lambda_c \) where \( \lambda_c = \sqrt{\frac{\sigma}{\Delta \rho g}} \) is the capillary length.

**Heuristic Argument:**

Change in Surface Energy:

\[
\Delta E_S = \sigma \cdot \int_0^\lambda ds = \sigma \left[ \int_0^\lambda ds - \lambda \right] = \frac{1}{2} \sigma \epsilon^2 k^2 \lambda.
\]

Change in gravitational potential energy:

\[
\Delta E_G = \int_0^\lambda -\frac{1}{2} \rho g (h^2 - h_0^2) dx = -\frac{1}{2} \rho g \epsilon^2 \lambda.
\]

When is the total energy decreased?

When \( \Delta E_{\text{total}} = \Delta E_S + \Delta E_G < 0 \), i.e. when \( \rho g > \sigma k^2 \), so \( \lambda > 2\pi \lambda_c \).

The system is thus unstable to long \( \lambda \).

**Note:**

1. The system is stabilized to small \( \lambda \) disturbances by \( \sigma \).
2. The system is always unstable for suff. large \( \lambda \).
3. In a finite container with width smaller than \( 2\pi \lambda_c \), the system may be stabilized by \( \sigma \).
4. System may be stabilized by temperature gradients since Marangoni flow acts to resist surface deformation. E.g., a fluid layer on the ceiling may be stabilized by heating the ceiling.

Figure 14.2: The base state and the perturbed state of the Rayleigh-Taylor system, heavy fluid over light.

Figure 14.3: Rayleigh-Taylor instability may be stabilized by a vertical temperature gradient.
14.2 Kelvin-Helmholtz Instability

We consider shear-driven instability of a gravitationally stable base state. Specifically, \( \rho_- \geq \rho_+ \), so the system is gravitationally stable, but destabilized by the shear.

Take \( k \) parallel to \( \mathbf{V} \), so \( (\mathbf{V} \cdot \mathbf{k})^2 = k^2 V^2 \) and the instability criterion becomes:

\[
\rho_- \rho_+ V^2 > (\rho_- - \rho_+) \frac{g}{k} + \sigma k \tag{14.17}
\]

Equivalently,

\[
\rho_- \rho_+ V^2 > (\rho_- - \rho_+) g \frac{\lambda}{2\pi} + \sigma \frac{2\pi}{\lambda} \tag{14.18}
\]

Note:

1. System stabilized to short \( \lambda \) disturbances by surface tension and to long \( \lambda \) by gravity.

2. For any given \( \lambda \) (or \( k \)), one can find a critical \( V \) that destabilizes the system.

**Marginal Stability Curve:**

\[
V(k) = \left( \frac{\rho_- - \rho_+}{\rho_- - \rho_+} \frac{g}{k} + \frac{1}{\rho_- - \rho_+} \sigma k \right)^{1/2} \tag{14.19}
\]

\( V(k) \) has a minimum where \( \frac{dV}{dk} = 0 \), i.e. \( \frac{1}{k} \frac{dV^2}{dk} = 0 \). This implies

\[
-\frac{\Delta \rho}{k^2} + \sigma = 0 \Rightarrow k_c = \sqrt{\frac{\Delta \rho}{\sigma}} = \frac{1}{V_c}
\]

The corresponding \( V_c = V(k_c) = \frac{2}{\rho_- - \rho_+} \sqrt{\Delta \rho g \sigma} \) is the minimal speed necessary for waves.

**E.g.** Air blowing over water: (cgs)

\[
V_c^2 = \frac{2}{12.3 \times 10^{-3}} \sqrt{1 \times 10^4 \times 70} \Rightarrow V_c \sim 650 \text{cm/s} \text{ is the minimum wind speed required to generate waves.}
\]

These waves have wavenumber \( k_c = \sqrt{\frac{1 \times 10^4}{70}} \approx 3.8 \text{ cm}^{-1} \), so \( \lambda_c = 1.6 \text{cm} \). They thus correspond to capillary waves.