4. Young’s Law with Applications

Young’s Law: what is the equilibrium contact angle \( \theta_e \)? Horizontal force balance at contact line:

\[
\cos \theta_e = \frac{\gamma_{SV} - \gamma_{SL}}{\gamma_{LV}} = 1 + \frac{S}{\gamma_{LV}} \quad \text{(Young 1805)}
\]

Note:

1. When \( S \geq 0 \), \( \cos \theta_e \geq 1 \) \( \Rightarrow \) \( \theta_e \) undefined and spreading results.

2. Vertical force balance not satisfied at contact line \( \Rightarrow \) dimpling of soft surfaces.
   E.g. bubbles in paint leave a circular rim.

3. The static contact angle need not take its equilibrium value \( \Rightarrow \) there is a finite range of possible static contact angles.

Back to Puddles: Total energy:

\[
E = (\gamma_{SL} - \gamma_{SV}) A + \gamma_{LV} A + \underbrace{\frac{1}{2} \rho g h^2 A}_{\text{grav. pot. energy}} = -S \frac{V}{h} + \frac{1}{2} \rho g V h
\]

Minimize energy w.r.t. \( h \):

\[
\frac{\partial E}{\partial h} = SV \frac{1}{h} + \frac{1}{2} \rho g V = 0 \quad \text{when} \quad -S/h^2 = \frac{1}{2} \rho g \quad \Rightarrow
\]

\[
h_0 = \sqrt{\frac{2S}{\rho g}} = 2 \ell_c \sin \frac{\theta_e}{2} \quad \text{gives puddle depth, where} \quad \ell_c = \sqrt{\sigma/\rho g}.
\]

Capillary Adhesion: Two wetted surfaces can stick together with great strength if \( \theta_e < \pi/2 \), e.g. Fig. 4.2.

Laplace Pressure:

\[
\Delta P = \sigma \left( \frac{1}{R} - \cos \frac{\theta_e}{2} \right) \approx -2\sigma \cos \frac{\theta_e}{2} \quad \text{i.e. low} \ P \ \text{inside film provided} \ \theta_e < \pi/2.
\]

If \( H \ll R \), \( F = \pi R^2 \frac{2\sigma \cos \theta_e}{H} \) is the attractive force between the plates.

E.g. for \( H_2O \), with \( R = 1 \ cm \), \( H = 5 \ \mu m \) and \( \theta_e = 0 \), one finds \( \Delta P \sim 1/3 \) atm and an adhesive force \( F \sim 10 N \), the weight of 1 l of \( H_2O \).

Note: Such capillary adhesion is used by beetles in nature.

4.1 Formal Development of Interfacial Flow Problems

Governing Equations: Navier-Stokes. An incompressible, homogeneous fluid of density \( \rho \) and viscosity \( \mu = \rho \nu \) (\( \mu \) is dynamic and \( \nu \) kinematic viscosity) acted upon by an external force per unit volume \( f \) evolves according to

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{(continuity)} \quad (4.3)
\]

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mathbf{f} + \mu \nabla^2 \mathbf{u} \quad \text{(Linear momentum conservation)} \quad (4.4)
\]
This is a system of 4 equations in 4 unknowns \((u_1, u_2, u_3, p)\). These N-S equations must be solved subject to appropriate BCs.

**Fluid-Solid BCs:** “No-slip”: \(u = U_{\text{solid}}\).

E.g. 1 Falling sphere: \(u = U\) on sphere surface, where \(U\) is the sphere velocity.

E.g. 2 Convection in a box: \(u = 0\) on the box surface.

But we are interested in flows dominated by interfacial effects. Here, in general, one must solve N-S equations in 2 domains, and match solutions together at the interface with appropriate BCs. Difficulty: These interfaces are free to move \(\Rightarrow\) Free boundary problems.

**Continuity of Velocity** at an interface requires that \(u = \hat{u}\).

And what about \(p\)? We’ve seen \(\Delta p \sim \sigma/R\) for a static bubble/drop, but to answer this question in general, we must develop stress conditions at a fluid-fluid interface.

Recall: **Stress Tensor.** The state of stress within an incompressible Newtonian fluid is described by the stress tensor: \(T = -pI + 2\mu E\) where \(E = \frac{1}{2} [\nabla u + (\nabla u)^T]\) is the deviatoric stress tensor. The associated hydrodynamic force per unit volume within the fluid is \(\nabla \cdot T\).

One may thus write N-S eqns in the form:

\[
\rho \frac{Du}{Dt} = \nabla \cdot T + f = -\nabla p + \mu \nabla^2 u + f.
\]

**Note:**

1. normal stresses (diagonals) \(T_{11}, T_{22}, T_{33}\) involve both \(p\) and \(u_i\)

2. tangential stresses (off-diagonals) \(T_{12}, T_{13}, \ldots\), involve only velocity gradients, i.e. viscous stresses

3. \(T_{ij}\) is symmetric (Newtonian fluids)

4. \(t(n) = n \cdot T =\) stress vector acting on a surface with normal \(n\)

**E.g. Shear flow.** Stress in lower boundary is tangential. Force / area on lower boundary:

\[
T_{yx} = \mu \frac{\partial u_y}{\partial x}\bigg|_{y=0} = \mu k
\]

is the force/area that acts on \(y\)-surface in \(x\)-direction.

**Note:** the form of \(T\) in arbitrary curvilinear coordinates is given in the Appendix of Batchelor.

**Figure 4.4:** E.g. 4 Water waves at an air-water interface.

**Figure 4.5:** Shear flow above a rigid lower boundary.