1 Discrete versus continuous steps in a random walk

1.1 Finding a generating function and $D_2(a)$

For $p_n = Ca^n$, the probability generating function is

$$P(z) = \sum_{n=-\infty}^{\infty} p_n z^n$$

$$= C \left[ \sum_{n=0}^{\infty} (az)^n + \sum_{n=1}^{\infty} \left( \frac{a}{z} \right)^n \right]$$

$$= C \left[ \frac{1}{1-az} + \frac{a}{z-a} \right] \quad (a < |z| < a^{-1})$$

$$= C \left[ \frac{(1-a^2)z}{(1-az)(z-a)} \right].$$

For normalization, $P(1) = \sum p_n = 1$, we need

$$C(a) = \frac{1-a}{1+a}$$

and thus

$$P(z) = \frac{(1-a^2)z}{(1-az)(z-a)}.$$}

The second moment is easily calculated as

$$\sigma^2 = \sum_{n=-\infty}^{\infty} n^2 p_n = \sum_{n=-\infty}^{\infty} n(n-1)p_n = P''(1)$$

where we use $\sum np_n = 0$ since $p_n = p_{-n}$. Since

$$P''(z) = 2aC[a(1-az)^{-3} + (z-a)^{-3}], \quad P''(1) = \frac{2(a+1)aC}{(1-a)^3}$$

we finally obtain the diffusivity,

$$D_2(a) = \frac{\sigma^2}{2\tau} = \frac{C(1+a)a}{(1-a)^3} = \frac{a}{(1-a)^2}$$

since the time step in the continuum approximation is $\tau = 1.$
1.2 Continuum approximation

Now we consider the continuum approximation,

\[ p(x) = \frac{1}{2b} e^{-|x|/b}, \quad (b = -1/\log a) \]

which has the same exponential decay as \( p_n \) for \( |n| \gg 1 \). The Fourier transform should look familiar (from problem set 2), but it’s easy enough to work out again:

\[
\hat{p}(k) = \int_{-\infty}^{\infty} e^{-ikx} p(x) dx = \frac{1}{2b} \left( \int_{-\infty}^{0} e^{-x(ik+1/b)} dx + \int_{0}^{\infty} e^{x(-ik+1/b)} dx \right) = \frac{1}{2} \left( \frac{1}{1 + ibk} + \frac{1}{1 - ibk} \right) = \frac{1}{1 + (bk)^2}.
\]

The cumulant generating function is

\[
\psi(k) = \log \hat{p}(k) = -\log(1 + (bk)^2) = \sum_{m=1}^{\infty} \frac{(-1)^m (bk)^{2m}}{m} = \sum_{n=1}^{\infty} \frac{(-i)^n c_n k^n}{n!}
\]

which implies \( c_{2m+1} = 0 \) and

\[
c_{2m} = \frac{(2m)! b^{2m}}{m}.
\]

The coefficients in the modified Kramers-Moyal expansion are then \( \bar{D}_{2m+1} = 0 \) and

\[
\bar{D}_{2m} = \frac{b^{2m}}{m} = \frac{1}{m(\log a)^{2m}}.
\]

1.3 Log-linear plot

The two diffusivities are

\[
D_2(a) = \frac{a}{(1-a)^2} \quad \text{and} \quad \bar{D}_2(b(a)) = \frac{1}{(\log a)^2}.
\]

In the limit \( a \to 1 \) the width of the distribution \( b = -1/\log a \) becomes much larger than the lattice spacing, and thus the continuum approximation should become exact, \( D_2 \sim \bar{D}_2 \), which is easily verified. In the opposite limit, \( a \to 0 \), the decay length for the distribution is much less than the lattice spacing, and the two models should be very different. In fact,

\[
\frac{\bar{D}_2}{D_2} \sim \frac{1}{a(\log a)^2} \to \infty \quad \text{as} \ a \to 0.
\]

These limits are also clear in figure 1.
2 First passage of \( N \) random walks in two dimensions

2.1 First passage position of a single walker

Since the diffusivity is a scalar (isotropic process), the Green function for the \( x \)-component (marginal probability density, after integrating out the \( y \)-component) will describe a one-dimensional \( x \) diffusion process with the same \( D \),

\[
G(x,t|0) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}.
\]

By symmetry, the \( x \) process plays no role in determining the first passage time, whose (Smirnov\(^1\)) probability density will be same as for a one-dimensional \( y \) diffusion process with the same \( D \),

\[
f(t|a) = -S'(t|a) = \frac{ae^{-a^2/4Dt}}{\sqrt{4\pi Dt^3}}
\]

where the survival probability is

\[
S(t|a) = \text{erf} \left( \frac{a}{\sqrt{4Dt}} \right).
\]

The hitting probability density can be calculated as

\[
\varepsilon(x|a) = \int_0^\infty f(t|a)G(x,t|0)dt
\]

since this is an integral over all times \( t \) of the probability that the \( x \)-component is \( x \) given that first passage occurs at time \( t \). As noted above, these events are independent, so the integrand is just a

\(^1\)There is a typo in the Exam 2, problem 2 solution from 2005: \( t \) should be \( t^3 \) under the square root. However, it is correct in Lecture 16 2005 notes and was correct in lecture this year.
The hitting probability of the first walker is given by the $x$ diffusion process sampled at this time,

$$
\varepsilon_N(x|a) = \int_0^\infty f_N(t|0) G(x,t|0). \tag{1}
$$

It does not seem that this integral can be performed analytically, so some numerical integrations are shown in figure 2.
The variance of the hitting position is given by
\[
\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \varepsilon_N(x|0) \, dx
\]
\[
= \int_0^{\infty} dt \, f_N(t|a) \int_{-\infty}^{\infty} dx \, x^2 G(x,t|0)
\]
\[
= \int_0^{\infty} dt \, f_N(t|a) 2Dt.
\]
For \( t \to \infty \) we have \( f(t|a) \propto t^{-3/2}, S(t) \propto t^{-1/2}, f_N(t|a) \propto t^{1-N/2} \), and thus the integrand decays like \( tf_N(t|a) \propto t^{N/2} \). Therefore, the variance is finite only if \( N/2 > 1 \Rightarrow N > 2 \Rightarrow N \geq 3 \). So, once again \( N_c = 3 \) is the magic number of walkers such that the first one will hit in a region of finite variance in space. This should come as no surprise, because this is the same critical number needed to have a finite mean first passage time for the first walker, as shown in lecture.

3 First passage to a circle

In the physical \( z \) plane, the walker is released at \( (x = a, y = 0) \) and hits the unit circle. We would like to map this domain with \( w = f(z) \) to the interior of the unit circle with the source at the origin in the mathematical \( w \) plane, where we know the complex potential is
\[
\Phi = \log w
\]
\[
\frac{2\pi}{2\pi}.
\]
We could choose a Möbius transformation with the constraints, \( f(a) = 0, f(1) = -1, f(-1) = 1 \), which yields
\[
f(z) = \frac{z-a}{az-1}.
\]
The complex potential in the \( z \) plane is therefore
\[
\Phi = \frac{1}{2\pi} \log \left( \frac{z-a}{az-1} \right) = \frac{1}{2\pi} \left( \log(z-a) - \log(z-a^{-1}) - \log a \right)
\]
which is clearly the sum of the source term and an image sink at \( (a^{-1}, 0) \) (and a constant). The hitting probability density is given by the normal electric field on the circle:
\[
\varepsilon(\theta|a) = \hat{n} \cdot \nabla \phi = -\text{Re}(e^{i\theta} \Phi)
\]
\[
= \text{Re} \left( \frac{1}{1 - a^{-1} e^{-i\theta}} - \frac{1}{1 - ae^{-i\theta}} \right)
\]
\[
= \frac{1}{2\pi} \left( \frac{1 - a^{-1} \cos \theta}{1 - 2a^{-1} \cos \theta + a^{-2}} - \frac{1 - a \cos \theta}{1 - 2a \cos \theta + a^2} \right).
\]
Therefore, the
\[
\frac{\varepsilon(0)}{\varepsilon(\pi)} = \left( \frac{a+1}{a-1} \right)^2
\]
which is nine for source at twice the radius \( (a = 2) \).

The geometrical interpretation follows from the cumulative distribution function
\[
\psi = \text{Im} \Phi = \frac{1}{2\pi} \left( \text{arg}(z-a) - \text{arg}(z-a^{-1}) \right) = \frac{\gamma}{\pi}
\]
where $\gamma$ is the angle formed at a point on the circle by drawing lines to the “charge” at $z = a$ and its “image” at $z = a^{-1}$. The probability of hitting between angle $\theta_1$ and $\theta_2$ on the circles is just the difference of two such angles, subtended from each of the points to $z = a$ and $z = a^{-1}$:

$$
\int_{\theta_1}^{\theta_2} \varepsilon(\theta|a) d\theta = \gamma_2 - \gamma_1.
$$

For infinitessimal $d\theta = \theta_2 - \theta_1$, we obtain the hitting probability density,

$$
\varepsilon(\theta|a) = \frac{1}{2\pi} \frac{d\gamma}{d\theta}.
$$